# LECTURE NOTES 9 - PART A

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## 1. Habit Formation Models

1.1. **Time-separability in utility function.** Recall that the fundamental asset pricing equation can be written as,

$$(risk\ premium)\,dt = -\left(\frac{dS}{S}\right)\left(\frac{dM}{M}\right). \tag{1.1}$$

We have shown that for time-separable utility functions (i.e., utility functions that only depend on current consumption) there exists some scalar/constant $\kappa$ such that the SDF in the underlying economy can be defined as,

$$\kappa \frac{\partial u}{\partial C}\left(C^*\left(t\right), t\right) = M\left(t\right),$$

where $C^*$ denotes optimal consumption.

For Breeden's CCAPM w/RA we have shown that the fundamental asset pricing equation is,

$$(risk\ premium)\,dt = CRRA \times \left(\frac{dS}{S}\right)\left(\frac{dC^*}{C^*}\right).$$

In Merton's ICAPM, consumption is driven by a vector $X$ of state variables related to the real business cycle, with value function (i.e., indirect utility function) $J\left(W, X, t\right)$ and intertemporal envelope condition,

$$\frac{\partial J}{\partial W} = \frac{\partial u}{\partial C}, \tag{1.2}$$

$$\Rightarrow \kappa J_W = M,$$

with $dM = \ldots dW + \ldots dX + \ldots dt$. Without time-separability (1) still holds, but (2) will not hold unless we assume that the state vector is Markov and then we still have,

$$\kappa J_W = M.$$

Why is $\kappa J_W$ a SDF?

Suppose we are given an extra unit of wealth $W$ at date $s$, and we want to consume and invest it in the risky asset $k$ (the tree in a Lucas type economy) until date $t$. Given the dynamic consumption-portfolio optimization problem then it must be,

$$J_W\left(W\left(s\right), X\left(s\right), s\right) = E_s\left[\left[\frac{S_k\left(t\right)}{S_k\left(s\right)} J_W\left(W\left(t\right), X\left(t\right), t\right)\right]\right],$$

$$\Rightarrow 1 = E_s\left[\left[\frac{S_k\left(t\right)}{S_k\left(s\right)} \frac{J_W\left(W\left(t\right), X\left(t\right), t\right)}{J_W\left(W\left(s\right), X\left(s\right), s\right)}\right]\right].$$

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1.2. **Constantinides (1990) internal habit consumption model.** Time-separability in the utility function is relaxed to allow adjacent complementarity in consumption or "habit persistence". This approach to consumption started from the seminal work of Duesenberry and others in consumption theory. In Constantinides' model, current utility is a function of past consumption and current consumption. So "habit persistence" is defined endogenously (i.e., is an internal process). Moreover, this setting is a simple example of Cox, Ingersoll, and Ross general equilibrium production economy model with assets' supply perfectly elastic (i.e., constant return technologies). The key role of "habit persistence" in asset pricing is to introduce a wedge between the intertemporal marginal rate of substitution (IMRS) and investors' coefficient of risk aversion (CRA) in order to resolve both Mehra-Prescott's equity risk premium and the risk-free rate puzzles.

(A1) $u\left(C, \underbrace{X}_{habit}, t\right) = exp\left(-\rho t\right) \frac{1}{1-\theta}\left(C - bX\right)^{1-\theta}$, for some constant $b$. If $b > 0$ (the special case assumed by Constantinides) the quantity $bX$ is the habit or "subsistence level" of consumption. If $b = 0$ then the utility function is of the standard time-separable form and displays constant relative risk aversion. Alternatively, if $b < 0$ then past consumption adds rather than subtracts from current utility, and the model is displaying "durability" in consumption. Note that $X$ is a function of past consumption with $X\left(0\right)$ as initial consumption: $dX = \left(-aX + C\right)dt$, and $X\left(t\right) = exp\left(-at\right)X\left(0\right) + \int_0^t exp\left(-a\left(t-s\right)\right)C\left(s\right)dS$, where $C\left(s\right)$ is an exponential weight average of past consumption.

The RA investor's dynamic consumption-portfolio problem is solved through the familiar Hamilton-Jacobi-Bellman (HJB) equation,

$$0 = \underbrace{Max}_{C,\pi} E\left[\int_0^\infty exp\left(-\rho t\right)u\left(C\left(t\right)\right)dt\right].$$

At date $t$ we have,

$$\underbrace{Max}_{C,\pi} E\left[\int_t^\infty exp\left(-\rho s\right)u\left(C\left(s\right)\right)ds\right],$$

given $W\left(t\right) = W$ and $X\left(t\right) = X$. Thus,

$$= exp\left(-\rho t\right) MaxE\left[\int_t^\infty exp\left(-\rho\left(s-t\right)\right)u\left(C\left(s\right)\right)ds\right],$$

given $W\left(t\right) = W$ and $X\left(t\right) = X$. And,

$$= exp\left(-\rho t\right) MaxE\left[\int_0^\infty exp\left(-\rho s\right)u\left(C\left(s\right)\right)ds\right],$$

given $W\left(0\right) = W$ and $X\left(0\right) = X$. Finally,

$$= exp\left(-\rho t\right)J\left(W, X\right).$$

Notice that time is not relevant given the infinite-horizon assumption. We rewrite,

$$= exp\left(-\rho t\right)W^{1-\theta}f\left(\frac{X}{W}\right),$$

where $f(z) = J(1, z)$. Then,

$$0 = \underbrace{Max}_{C,\pi} \left\{ exp(-\rho t)\, u(C, X) - \underbrace{\rho exp(-\rho t)\, J(W, X)}_{J_t} + \right.$$

$$+ exp(-\rho t)\, J_W \left[ rW + W\pi(\mu - r) + \underbrace{W\pi\sigma dB}_{-C} \right] +$$

$$\left. + exp(-\rho t)\, J_X \underbrace{dX}_{(-aX+C)dt} + \frac{1}{2} exp(-\rho t)\, J_{WW} W^2 \pi^2 \sigma^2 \right\}. \tag{1.3}$$

Because habit persistence is endogenous i.e., $dX$ is a function of past consumption only, we don't have second order terms in the HJB involving $dX$. The F.O.N.C. w.r.t. $C$ is,

$$u_C - J_W + J_X = 0,$$

$$J_W = u_C + J_X, \tag{1.4}$$

where if $b$ is positive then the sign of $J_X$ is negative. One needs to know what is $J$. The F.O.N.C. w.r.t. $\pi$ is,

$$J_W(\mu - r)\, W + J_{WW} W^2 \sigma^2 \pi = 0,$$

$$\pi^* = \frac{\mu - r}{\frac{-W j_{WW}}{J_W} \sigma^2} = (CRA\ in\ terms\ of\ J)^{-1} \frac{\mu - r}{\sigma^2}. \tag{1.5}$$

1.2.1. *Empirical evidence of internal habit consumption.* Heaton studied the empirical capabilities of the internal habit formation model assuming that consumption and dividend growth rates follow a bivariate VAR with parameters estimated using simulated method of moments (that is to accomodate time aggregation). He found evidence of local durability of consumption and significant habit formation in the long-run.

Assuming local durability and habit persistence in the long-run improves the fit of the CCAPM compared to both the time-additive model of Breeden and the special case of pure habit persistence (without local durability). This improvement though comes with a significant cost in terms of relative high volatility in the SDF.

1.3. **Campbell and Cochrane (1999) external habit consumption model.** Campbell and Cochrane introduced a model of external habit preference or "keeping with the Joneses" preferences. The model is an example of a Lucas pure-exchange economy (unlike Constantinides that is an example of a production economy) with perfectly inelastic supplied assets (i.e., the trees of the Lucas economy). Consequently, in the external habit model $dX$ has a $dB$ term that entails second order terms in the HJB equation and is not dependent on $C$ i.e., the habit is an exogenous stochastic process.

(A1) $u\left(C, \underbrace{X}_{habit}, t\right) = exp(-\rho t)\, \frac{1}{1-\theta} (C - X)^{1-\theta}$. Now $X$ is exogenous with $\frac{dX}{X} = something\ dt + \phi\sigma_C dB$, for $0 < \phi < 1$.

(A2) $\frac{dC}{C} = \mu_C dt + \sigma_C dB_C,$

(A3) $\frac{dS}{S} = \mu_S dt + \sigma_S dB_S$, and
(A4) $dB_C, dB_S$ have correlation $\rho$.

Note,

$$\frac{\partial u}{\partial C} = (C - X)^{-\theta},$$

$$\frac{\partial^2 u}{\partial C^2} = -\theta (C - X)^{-\theta - 1},$$

$$\frac{\partial^2 u}{\partial C \partial X} = \theta (C - X)^{-\theta - 1}.$$

Recall,

$$\kappa \frac{\partial u}{\partial C} = M$$

Then by Ito's lemma,

$$dM = \kappa \frac{\partial^2 u}{\partial C^2} dC + \kappa \frac{\partial^2 u}{\partial C \partial X} dX + \ldots dt,$$

$$\Rightarrow \frac{dM}{M} = \frac{\frac{\partial^2 u}{\partial C^2}}{\frac{\partial u}{\partial C}} dC + \frac{\frac{\partial^2 u}{\partial C \partial X}}{\frac{\partial u}{\partial C}} dX + \ldots dt$$

$$= -\frac{\theta}{C - X} dC + \frac{\theta}{C - X} dX + \ldots dt$$

$$= -\theta \frac{d (C - X)}{C - X} + \ldots dt,$$

$$\Rightarrow (risk\ premium)\, dt = \theta \left( \frac{dS}{S} \right) \left( \frac{d (C - X)}{C - X} \right). \tag{1.6}$$

Notice that "consumption surplus" (i.e., $C - X$) is low during economic recessions and high during economic booms. Moreover, volatility is inversely proportional to $C - X$ too. That is, is low during economic booms and high during economic recessions. Given $dC$ and $dX$ we write,

$$dC - dX = something\ dt + [\phi \sigma_C (C - X) + (1 - \phi) C \sigma_C] dB_C,$$

$$\Rightarrow \frac{d (C - X)}{C - X} = \left[ \phi \sigma_C + (1 - \phi) \frac{C}{C - X} \sigma_C \right] dB_C + something\ dt,$$

$$(risk\ premium)\, dt = \theta \left[ \phi \sigma_C + (1 - \phi) \frac{C}{C - X} \sigma_C \right] \sigma_S \rho$$

$$\Rightarrow (risk\ premium)\, dt = \underbrace{\theta \sigma_C \sigma_S \rho}_{time\ separable\ part} \left[ \phi + (1 - \phi) \frac{C}{C - X} \right]. \tag{1.7}$$

*Case* 1. Economic recessions: $\frac{C}{C-X}$ will be large as $C - X$ is small (eventually of small order) during economic recessions.

*Case* 2. Economic booms: $\frac{C}{C-X}$ will be small as $C - X$ is large during economic booms.

1.3.1. *Equity risk premium puzzle resolution.* Recall that in the time-separable case the Hansen Jagannathan lower bound is ,

$$|risk\ premium| = |\theta\sigma_C\sigma_S\rho| \leq \theta\sigma_C\sigma_S,$$
$$\Rightarrow |Sharpe's\ ratio| \leq \theta\sigma_C.$$

Because $\sigma_C$ is historically too low, then $\theta$ has to be unrealistically high. Under habit consumption though,

$$|risk\ premium| = \left|\theta\sigma_C\sigma_S\rho\left[\phi + (1-\phi)\frac{C}{C-X}\right]\right| \leq \theta\sigma_C\sigma_S\left[\phi + (1-\phi)\frac{C}{C-X}\right],$$

$$\Rightarrow |Sharpe's\ ratio| \leq \theta\sigma_C\left[\phi + (1-\phi)\frac{C}{C-X}\right]. \tag{1.8}$$

Now $\theta$ doesn't have to be unrealistically high to satisy (8) as $\left[\phi + (1-\phi)\frac{C}{C-X}\right]$ can be large (especially during economic recessions).

1.3.2. *Risk free rate puzzle resolution.* Under time-separable utility functions the risk free rate is equal to,

$$r = \rho + \theta\mu_C - \underbrace{\frac{1}{2}\theta(1-\theta)\sigma_C^2,}_{o(\cdot)}$$

where $r$ is too high w.r.t. the historic observed level given that the third term in the equation is of small order (i.e., $\sigma_C$ is too low). Under habit consumption though,

$$r = \rho + \theta\mu_C - \frac{1}{2}\theta(1-\theta)\sigma_C^2\left[\phi + (1-\phi)\frac{C}{C-X}\right], \tag{1.9}$$

where $r$ is no longer too high given that the third term in (9) can be large (especially during economic recessions).

1.3.3. *Empirical evidence of external habit consumption.* A formal econometric implementation of the external habit formation model is undertaken in Bansal et al. (2004), where it is assumed that dividends are co-integrated with consumption (i.e., both share a common stochastic linear trend). They estimate the model using a simulated method of moments over an auxiliary VAR. Using data from 1929-2001 they found that the external habit model does a good job in matching the moments of consumption and dividend growth. However, this model still does not resolve the volatility puzzle: the price/dividend ratio is more volatile historically than what is implied by habit formation.

## 2. Recursive Utility Models

2.1. **Kreps and Porteus (1978) model.** Another class of time-inseparable utility is the one developed by Kreps and Porteus in discrete time, and Duffie and Epstein (1992) in continuous time. The specification is recursive as lifetime utility denoted by $V_t$ depends on expected values of future lifetime utility $V_s, s > t$ (i.e., forward looking model with conditional certainty equivalents CE). The future is summarized by a single index: the certainty equivalent of next period's utility.

Recall that for a myopic (i.e., one time period from 0 to 1) investor,

$$u = u(C_0) + \delta u(C_1),$$

where the slope of the indifference curve is the intertemporal marginal rate of substitution of consumption $IMRS = \left.\frac{dC_1}{dC_0}\right|_{u=constant} = \frac{u'(C_0)}{\delta u'(C_1)}$. In this special case, the elasticity of the IMRS (EIS) is equal to the investor's coefficient of risk aversion,

$$\frac{\partial log IMRS}{\partial log C_1} = CRA.$$

For an infinite horizon investor we have,

$$V_0 = E_0\left[\sum_{i=0}^{\infty} f\left(C_i, V_{i+1}\right)\right],$$

where $f$ is some aggregator function of the consumption plan $C_0, C_1, \ldots$ and lifetime utility $V_0, V_1, \ldots$. The aggregator function has two components: 1) a risk component that characterizes the (intra-temporal) trade-off across the outcomes of a static gamble, which defines the certainty equivalent of future utility; and 2) a time component that characterizes the (inter-temporal) trade-off between current consumption and the certainty equivalent of lifetime utility. In general,

$$(\forall i) \quad V_i = E_i\left[\sum_{j=i}^{\infty} f\left(C_j, V_{j+i}\right)\right]. \tag{2.1}$$

Consider first the time-separable case $f(C, V) = u(C) - (1-\delta)V$. Then,

$$V_0 = E_0\left[u\left(C_0\right) - (1-\delta)V_1 + f\left(C_1, V_2\right) + f\left(C_2, V_3\right) + \cdots\right],$$

$$V_1 = E_1\left[u\left(C_1\right) - (1-\delta)V_2 + f\left(C_2, V_3\right) + f\left(C_3, V_4\right) + \cdots\right],$$

$$\Rightarrow V_0 = E_0\left[u\left(C_0\right) - (1-\delta)\left\{E_1\left[u\left(C_1\right) - (1-\delta)V_2 + f\left(C_2, V_3\right) + f\left(C_3, V_4\right) + \cdots\right]\right\} + \right.$$
$$\left. + f\left(C_1, V_2\right) + f\left(C_2, V_3\right) + \cdots\right]$$
$$= E_0\left[u\left(C_0\right) - (1-\delta)u\left(C_1\right) + (1-\delta)^2 V_2 - (1-\delta)\left[f\left(C_2, V_3\right) + f\left(C_3, V_4\right) + \cdots\right] + ,\right.$$
$$+u\left(C_1\right) - (1-\delta)V_2 + f\left(C_2, V_3\right) + f\left(C_3, V_4\right) + \cdots, by\ law\ of\ iterated\ expectations,$$
$$= E_0\left[u\left(C_0\right) + \delta\left(C_1\right) + \cdots\right],$$

which is a special case of recursive preferences. Without time-separability,

$$V_0 = E_0\left[\sum_{i=0}^{\infty} f\left(C_i, V_{i+1}\right)\right] = E_0\left[f\left(C_0, V_1\right) + f\left(C_1, V_2\right) + \cdots\right] = E_0\left[f\left(C_0, V_1\right) + V_1\right], \tag{2.2}$$

$$\Rightarrow IMRS = -\frac{\frac{\partial V_0}{\partial C_0}}{\frac{\partial V_0}{\partial C_1}},$$

where $C_0$ enters through $f(C_0, V_1)$, and $C_1$ enters through $f(C_1, V_2)$ and $V_1$ in $f(C_0, V_1)$. Recall that $V_1 = E_1\left[f\left(C_1, V_2\right) + f\left(C_2, V_3\right) + \cdots\right]$. Thus,

$$\frac{\partial V_0}{\partial C_0} = \frac{\partial f\left(C_0, V_1\right)}{\partial C_0} = u'\left(C_0\right), and$$

$$\frac{\partial V_0}{\partial C_1} = \frac{\partial f\left(C_1, V_2\right)}{\partial C_1} + \frac{\partial f\left(C_0, V_1\right)}{\partial V_1} + \frac{\partial V_1}{\partial C_1}$$
$$= \frac{\partial f\left(C_1, V_2\right)}{\partial C_1} + \frac{\partial f\left(C_0, V_1\right)}{\partial V_1}\frac{\partial f\left(C_1, V_2\right)}{\partial C_1}$$

$$= \underbrace{\left(1 + \frac{\partial f\left(C_0, V_1\right)}{\partial V_1}\right)}_{\delta} \underbrace{\left(\frac{\partial f\left(C_1, V_2\right)}{\partial C_1}\right)}_{u'(C_1)}, and$$

$$IMRS = \frac{\frac{\partial f(C_0,V_1)}{\partial C_0}}{\left(1 + \frac{\partial f(C_0,V_1)}{\partial V_1}\right)\left(\frac{\partial f(C_1,V_2)}{\partial C_1}\right)}.$$

Recall that the CRA is equal to $-\frac{V_1 \frac{\partial^2 V_0}{\partial V_1^2}}{\frac{\partial V_0}{\partial V_1}}$, and $\frac{\partial V_0}{\partial V_1} = 1 + \frac{\partial f(C_0,V_1)}{\partial V_1}$, $\frac{\partial^2 V_0}{\partial V_1^2} = \frac{\partial^2 f(C_0,V_1)}{\partial V_1^2}$. Then,

$$CRA = \frac{V_1\left(\frac{\partial^2 f(C_0,V_1)}{\partial V_1^2}\right)}{\left(1 + \frac{\partial f(C_0,V_1)}{\partial V_1}\right)} \neq elasticity\ of\ IMRS. \tag{2.3}$$

Which makes sense, as the CRA proxies for investors' willingness to substitute consumption across states of nature at any given date $t$ (an intra-temporal measure), and the EIS captures investors' willingness to substitute consumption over time in response to changing economic conditions (an inter-temporal measure). Hence, investors with recursive preferences care about the timing of the resolution of uncertainty (i.e., investors are no longer standard expected utility maximizers).

2.1.1. *The Epstein-Zin-Weil specification.* Following Epstein and Zin (1989) and Weil (1989) agents maximize a recursive utility function with aggregator function,

$$f\left(C, V\right) = \nu^{-1}\left[\left(1-\delta\right)\nu\left(C\right) + \delta\nu\left(V\right)\right],$$

where $\nu\left(x\right) = \frac{x^{1-\theta}-1}{1-\theta}$; $\theta \equiv \frac{\gamma}{1-\psi}$ (with $\nu\left(x\right) = log\left(x\right)$if $\theta = 1$); $1-\gamma$ is the CRA; and $\psi$ is the EIS. The SDF of the underlying economy is,

$$\delta^{\theta} E\left[\left\{\frac{C_{t+1}}{C_t}\right\}^{-\left(\frac{\theta}{\psi}\right)} r_{M,t+1}^{\theta-1} r_{t+1} \left|\mathcal{F}_t\right.\right] = 1, \tag{2.4}$$

where $r_{M,t+1}^{\theta-1}$ is the one-period holding period return on the wealth portfolio; and $r_{t+1}$ is the one-period holding period return of any security in the mean-variance efficient set.

*Case* 1. $\gamma < (1-\psi)$, and an early resolution of uncertainty is preferred.

*Case* 2. $\gamma > (1-\psi)$, and a later resolution of uncertainty is preferred.

Bansal and Yaron (2004) introduce a persistent component in consumption and dividend growth in an asset pricing model with Epstein-Zin-Weil preferences. Assuming $\psi > 1$ and $\gamma = 10$, the model generates an equity risk premium consistent with its historical value. Critical to the success of the model is the balancing of intertemporal substitution and risk aversion with $\gamma \neq \frac{1}{\psi}$ and $\psi > 1$.

## References

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