LECTURE NOTES 3

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I. Markowitz-Tobin Mean-Variance Portfolio Analysis

Assumption $\Rightarrow$Mean-Variance preferences (Markowitz, 1952)

1. Quadratic utility function,

$$E\left[\tilde{w}-b\tilde{w}^{2}\right]=E\left[\tilde{w}\right]-b\left\{Var\left(\tilde{w}\right)+\left(E\left[\tilde{w}\right]\right)^{2}\right\}\ by\ definition\ of\ variance. \tag{1}$$

2. Normally distributed asset returns.

Assume k risky assets with,

$$\tilde{R}=\begin{pmatrix}\tilde{R}_1\\ \vdots\\ \tilde{R}_K\end{pmatrix},\quad vector\ of\ gross\ returns\ (\iota+\tilde{r}),$$

$$\bar{R}=\begin{pmatrix}\bar{R}_1\\ \vdots\\ \bar{R}_K\end{pmatrix},\quad vector\ of\ expected\ returns,$$

$$\iota=\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix},\quad vector\ of\ K\ ones,$$

$$w=\begin{pmatrix}w_1\\ \vdots\\ w_K\end{pmatrix},\quad vector\ of\ portfolio\ weights\ (a\ function\ of\ wealth\ invested\ in\ each\ asset),$$

$$\bar{R}_p=\left\{\begin{array}{c} w^{\top}\bar{R}\\ w^{\top}\bar{R}+\left(1-\iota^{\top}w\right)R_f\end{array}\right.,$$

where $R_f=1+r_f$. If there is no riskless asset, then we require $\iota^{\top}w=1$, otherwise $\left(1-\iota^{\top}w\right)=$ fraction of wealth invested in the riskless asset (could be negative i.e., short position). Define,

$$V=E\left[\left(\tilde{R}-\bar{R}\right)\left(\tilde{R}-\bar{R}\right)^{\top}\right]=\begin{pmatrix}var\,1 & \cdots & cov\,1\&K\\ \vdots & \ddots & \vdots\\ cov\,K\&1 & \cdots & var\,K\end{pmatrix} \tag{2}$$

as the variance-covariance matrix of the k risky assets. Also,

$$\tilde{R}_p=\left\{\begin{array}{c} w^{\top}\tilde{R}\\ w^{\top}\tilde{R}+\left(1-\iota^{\top}w\right)R_f\end{array}\right..$$

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And,

$$\sigma_p^2 = E\left[\left(\tilde{R}_p - \bar{R}_p\right)^2\right]$$

$$= E\left[\left(w^\top \tilde{R} - w^\top \bar{R}\right)^2\right]$$

$$= E\left[\left\{w^\top\left(\tilde{R} - \bar{R}\right)\right\}^2\right] = E\left[w^\top\left(\tilde{R} - \bar{R}\right)\left(\tilde{R} - \bar{R}\right)^\top w\right]$$

$$= w^\top E\left[\left(\tilde{R} - \bar{R}\right)\left(\tilde{R} - \bar{R}\right)^\top\right] w$$

$$= w^\top V w.$$

Note that V is symmetric and p.s.d. (positive semi-definite).

Claim. If it is not definite, then there is no risk-free portfolio and no perfect hedge in the economy.

Proof. By example. Let $K = 2$

$V = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{pmatrix}$,

where $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$ is the correlation coefficient. Note that there exists a zero-risk portfolio iff $\rho = \pm 1$ (**Why?**).

$Var\tilde{R}_p = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho\sigma_1\sigma_2$.

If $\rho = 1$ then $Var\tilde{R}_p = \left(w_1\sigma_1 + w_2\sigma_2\right)^2 = 0$ if $w_1 = \frac{-w_2\sigma_2}{\sigma_1}$.

If $\rho = -1$ then $Var\tilde{R}_p = \left(w_1\sigma_1 - w_2\sigma_2\right)^2 = 0$ if $w_1 = \frac{w_2\sigma_2}{\sigma_1}$. □

1. The Portfolio Choice Problem. Find the portfolio with minimum variance for a given $\bar{R}_p$.

$$\underset{w}{Min}\frac{1}{2}w^\top V w \tag{3}$$

$$s.t. \quad w^\top \bar{R} + \left(1 - \iota^\top w\right) R_f = \bar{R}_p.$$

Assuming an interior solution exists, the Lagrangian function can be written as,

$$\mathcal{L} \equiv \frac{1}{2}w^\top V w - \lambda\left(w^\top \bar{R} + \left(1 - \iota^\top w\right) R_f - \bar{R}_p\right),$$

with first order necessary conditions (F.O.N.C.):

1. w.r.t. w

$\frac{\partial \mathcal{L}}{\partial w} \equiv \underbrace{\frac{1}{2}2}_{derivative\ of\ matrix} w^\top V - \lambda\left(\bar{R}^\top - R_f\iota^\top\right) = 0 \Longleftrightarrow Vw - \lambda\left(\bar{R} - R_f\iota\right) = 0 \Rightarrow$

$w^* = \lambda V^{-1}\left(\bar{R} - R_f\iota\right)$, where V has full rank.

2. w.r.t. λ

$\left(\bar{R} - R_f\iota\right)^\top w = \bar{R}_p - R_f \Rightarrow \lambda\left(\bar{R} - R_f\iota\right)^\top V^{-1}\left(\bar{R} - R_f\iota\right) = \bar{R}_p - R_f$, after substituting the optimal w^*,

$\Rightarrow \lambda = \frac{\bar{R}_p - R_f}{\left(\bar{R} - R_f \iota\right)^\top V^{-1}\left(\bar{R} - R_f \iota\right)}$, where $\left(\bar{R} - R_f \iota\right)$is the vector of risk premiums.

$$\Rightarrow w^* = \underbrace{\frac{\bar{R}_p - R_f}{\left(\bar{R} - R_f \iota\right)^\top V^{-1}\left(\bar{R} - R_f \iota\right)}}_{scalar} \underbrace{V^{-1}\left(\bar{R} - R_f \iota\right)}_{k \times 1}. \tag{4}$$

$$\Leftrightarrow \underbrace{\frac{\left(\bar{R} - R_f \iota\right)^\top V^{-1}\left(\bar{R} - R_f \iota\right)}{\bar{R}_p - R_f}}_{scalar} V w^* = \bar{R} - R_f \iota \Rightarrow \eta \cdot w^T V w^* = w^T\left(\bar{R} - R_f \iota\right) \Rightarrow$$

$\bar{R}_k - R_f = \eta \cdot cov\ of\ asset\ k\ with\ efficient\ portfolio\ \forall k$.

$\Leftrightarrow \sigma_p = \sqrt{\frac{\left(\bar{R}_p - R_f\right)}{\eta}}$.

If $\bar{R}_p = R_f \Rightarrow \sigma_p = 0$, if $\bar{R}_p > R_f \Rightarrow \sigma_p \uparrow$ directly proportional to the portfolio's risk premium (The CML). The risk-return tradeoff is linear and positive.

Definition 1. (Sharpe's ratio) We define the Sharpe Ratio as $\frac{risk\ premium}{stdev} = \frac{\bar{R}_p - R_f}{\sigma_p}$. That is, the equilibrium excess return on the efficient portfolio per unit of portfolio risk i.e., the slope of the CML.

Claim. The efficient (tangency) portfolio is the one that maximizes Sharpe's ratio.

Proof. Follows from the previous analysis. □

II. CAPM

The Markowitz-Tobin mean-variance analysis is normative as it prescribes the best way to allocate wealth among various assets. Alternatively, we can interpret it as a positive or descriptive theory of what investors actually do. Under this interpretation, we can extend the portfolio problem to that of asset pricing under equilibrium.

Note that the mean-variance analysis does not assume a RA with SI or TI utility function and initial (aggregate) endowment/wealth. It does assume that investors have the same expectations regarding the probability distributions of asset returns, all assets are tradable, there are no indivisibilites in asset holdings, and there are no limits to borrowing and lending at the risk-free rate of return.

Let w_i =portfolio of investor i and $\varsigma_i = \frac{wealth\ of\ investor\ i}{total\ wealth}$ investor's i fraction of total wealth. Then, the market portfolio is $w_m = \sum_i \varsigma_i w_i\ \forall i \in I$. This market portfolio is a claim to the total (future) endowment of the economy regardless of the state of the world. By holding a share of the market portfolio, the investor assures a fraction ς_i of total future endowment almost sure. Thus, the equilibrium relationship (4) can be interpreted as a relation between expected excess returns on any asset and the expected excess return on the (broad) market portfolio with gross return $\tilde{R}_m$ proxying for systematic risk.

Assume $\exists$ riskless asset. Define $\tilde{X}_k = \tilde{R}_k - R_f$ as the excess return on asset k, so $\bar{X}_k = \bar{R}_k - R_f$ is the risk premium on asset k and $\tilde{X}_m = \tilde{R}_m - R_f$ is the market excess return. The behavioral problem of the ith investor is now,

$$= \underset{w}{Min} \frac{1}{2} w^\top V w, \tag{5}$$

$$s.t. \quad w^{\top}\bar{X} = \bar{X}_m.$$

Assuming an interior solution exists, the Lagrangian is $\mathcal{L} \equiv \frac{1}{2}w^{\top}Vw - \lambda\left(w^{\top}\bar{X} - \bar{X}_m\right)$, where the F.O.N.C. are,

1. w.r.t. w,

$\frac{\partial\mathcal{L}}{\partial w} \equiv Vw - \lambda\bar{X} = 0$, and

2. w.r.t. λ,

$w^{\top}\bar{X} - \bar{X}_m = 0$,

$\Rightarrow w^{\top}\bar{X} = \bar{X}_m$ and $\lambda V^{-1}\bar{X} = w$. Substituting the latter in the former gives $\lambda\bar{X}^{\top}V^{-1}\bar{X} = \bar{X}_m \Rightarrow \lambda = \frac{\bar{X}_m}{\bar{X}^{\top}V^{-1}\bar{X}}$

$\Rightarrow w^{*} = \frac{\bar{X}_m}{\bar{X}^{\top}V^{-1}\bar{X}} \cdot V^{-1}\bar{X}$. Rearranging terms and multiplying by w^{T}on both sides of the equal sign as before leads to,

$$(\forall k) \quad \bar{R}_k - R_f = \eta \cdot cov\left(\tilde{R}_k, \tilde{R}_m\right) \tag{6}$$

$$where \quad \tilde{R}_m = w^{\top}\tilde{R} + \left(1 - \iota^{\top}w\right)R_f.$$

And the standard deviation of the market (mean variance efficient) portfolio is $\lambda \times \sqrt{\bar{X}^{\top}V^{-1}\bar{X}}$.[1] The Sharpe ratio is $\sqrt{\bar{X}^{\top}V^{-1}\bar{X}}$ by definition (i.e., the slope of the CML that now passes through the origin) and $\iota^{\top}w^{*} = 1$ (Two-fund allocation i.e., investors are long on the market portfolio consisting only of risky assets and short on the risk free asset). The Sharpe ratio can be interpreted now as the market price of systematic or non-diversifiable risk. Recall that $\eta = \frac{\bar{X}_m}{var\left(\tilde{R}_m\right)}$. Thus, substituting the latter in (6) gives the CAPM or SML,

$$\bar{R}_k - R_f = \underbrace{\frac{cov\left(\tilde{R}_k, \tilde{R}_m\right)}{\sigma_m^2}}_{\beta_k}\left(\bar{R}_m - R_f\right)\forall k. \tag{7}$$

Equation (7) shows that the required excess expected return for a risky asset is a linear function of systematic risk with the price of systematic risk as slope. Also

2. Empirical Test of the CAPM. Consider an orthogonal projection of $\tilde{R}_k$ on $\tilde{R}_p$ where p is some portfolio proxying for the market portfolio (The empirical SML). Then write,

$$\tilde{R}_k = a_k + b_k\tilde{R}_p + \tilde{\varepsilon}_k, \forall k. \tag{8}$$

where $E\left[\hat{\varepsilon}_k\right] = 0$ and $E\left[\hat{\varepsilon}_k, \tilde{R}_p\right] = 0$. Note $\bar{R}_k = a_k + b_k\bar{R}_p$

$\Rightarrow E\left[\left(\tilde{R}_k - a_k - b_k\tilde{R}_p\right), \tilde{R}_p\right] = 0 \Rightarrow cov\left(\tilde{R}_k, \tilde{R}_p\right) - b_k var\left(\tilde{R}_p\right) = 0 \Rightarrow b_k = \frac{cov\left(\tilde{R}_k, \tilde{R}_p\right)}{var\left(\tilde{R}_p\right)}$ and substituting

$\Rightarrow a_k = \bar{R}_k - \frac{cov\left(\tilde{R}_k, \tilde{R}_p\right)}{var\left(\tilde{R}_p\right)}\bar{R}_p$.

Condition. If $\tilde{R}_p$ is minimum variance (mv)-efficient then $a_k = 0\ \forall k$.

[1] From (6), the standard deviation of the market portfolio return is $Var\left(\tilde{R}_m\right) = \frac{\bar{X}_m}{\eta}$. From the F.O.N.C. w.r.t. λ, we know that $\bar{X}_m = \lambda \cdot \left(\bar{X}^{\top}V^{-1}\bar{X}\right)$ and $Var\left(\tilde{R}_m\right) = \frac{\lambda \cdot \left(\bar{X}^{\top}V^{-1}\bar{X}\right)}{\eta}$. Notice that $\lambda = \frac{1}{\eta}$ and $Var\left(\tilde{R}_m\right) = \lambda^2 \cdot \left(\bar{X}^{\top}V^{-1}\bar{X}\right)$.

Remark. The test using the SML is a test of efficient mean-variance preferences not the CAPM.

3. Roll's Critique.

i. If the market is assumed to be ex ante mean variance efficient then the CAPM holds exactly.

ii. As the return of the market portfolio is unobservable, the best we can do is an ex post test of mean variance efficiency on some index portfolio we choose to proxy for the market portfolio, without actually testing the CAPM.

III. Black's Zero-Beta CAPM

Pick any mv-efficient portfolio w. Then, $\exists\, \alpha$ s.t. $\bar{R}_k - \alpha = \frac{cov\left(\tilde{R}_k, \tilde{R}_p\right)}{\sigma_p^2}\left(\bar{R}_p - \alpha\right) \forall k$, where α is the expected return of any portfolio or asset with zero covariance with the mv-efficient portfolio.

Recall that the projection of $\tilde{R}_k$ on $\tilde{R}_p$ is $\tilde{R}_k = a_k + \frac{cov\left(\tilde{R}_k, \tilde{R}_p\right)}{var\left(\tilde{R}_p\right)}\tilde{R}_p + \tilde{\varepsilon}_k$. Notice that $\bar{R}_k = \alpha - b_k\alpha + b_k\bar{R}_p$ if $b_k = \frac{cov\left(\tilde{R}_k, \tilde{R}_p\right)}{var\left(\tilde{R}_p\right)}$. Thus, the restriction to be tested in the zero-beta CAPM is now $a_k = (1 - b_k)\,\alpha = 0 \Rightarrow \underbrace{\frac{a_1}{1-b_1} = \frac{a_2}{1-b_2} = \cdots = \frac{a_n}{1-b_n}}_{n-1\ restrictions}.$

IV. Arbitrage Pricing Theorem

We assume N risk factors and K assets in the economy such that $K > N$.

Definition 2. (**Risk factor**) $\tilde{f}_n$ is the random realization of the nth risk factor.

Definition 3. (**Factor sensitivity**) b_{kn} is the sensitivity of the kth asset to the nth risk factor.

Definition 4. (**Idiosyncratic risk**) $\tilde{\varepsilon}_k$ is the idiosyncratic risk specific to asset k.

(A1) $E\left[\tilde{f}_n\right] = 0 \quad \forall n = 1, \ldots, N$. (Normalization - mean = 0).

(A2) $E\left[\tilde{f}_n, \tilde{f}_m\right] = 0 \quad \forall n, m = 1, \ldots, N$ and $n \neq m$ (Risk factors are mutually independent).

(A3) $E\left[\tilde{f}_n^2\right] = 1 \quad \forall n = 1, \ldots, N$. (Normalization - variance = 1).

(A4) $E\left[\tilde{\varepsilon}_k\right] = 0 \quad \forall k = 1, \ldots, K$.

(A5) $E\left[\tilde{\varepsilon}_k, \tilde{\varepsilon}_j\right] = 0 \quad \forall k, j = 1, \ldots, K$ and $k \neq j$.

(A6) $E\left[\tilde{\varepsilon}_k, \tilde{f}_n\right] = 0 \quad \forall k = 1, \ldots, K$ and $\forall n = 1, \ldots, N$.

(A7) $E\left[\tilde{\varepsilon}_k^2\right] < \infty \quad \forall k = 1, \ldots, K$.

Normalizations (A1)-(A3) can be weakened as we can always transform by linear combination the risk factors to make them satisfy such conditions. If we define the expected return of asset k as a_k then we have the linear projection,

$$\tilde{R}_k = a_k + \sum_{n=1}^{N} b_{kn}\tilde{f}_n + \tilde{\varepsilon}_k \quad \forall k = 1, \ldots, K \ and\ \forall n = 1, \ldots, N. \tag{9}$$

Definition 5. (**Asymptotic arbitrage**) Let a portfolio of K assets be described by the vector of weights $w^K = \left(w_1^K \; w_2^K \cdots w_K^K\right)^\top$. Consider the increasing sequence $K = 2, 3, \ldots$ Let σ_{kj} be the covariance between the return of asset k and return of asset j. An asymptotic arbitrage exists if the following conditions hold:

(A) $\sum_{k=1}^K w_k^K = 0$ (zero net investment).

(B) $\lim\limits_{K\to\infty} \sum_{k=1}^K \sum_{j=1}^K w_k^K w_j^K \sigma_{kj} = 0$ (Portfolio's return becomes certain).

(C) $\sum_{k=1}^K w_k^K a_k > 0$ (portfolio's return bounded above zero).

Theorem 6. *(**APT**) If $\nexists$ asymptotic arbitrages then,*

$$a_k = \underbrace{\lambda_0}_{constant} + \sum_{n=1}^N b_{kn} \underbrace{\lambda_n}_{risk\ premium\ for\ \tilde{f}_n} + \underbrace{v_k}_{expected\ return\ deviation}, \tag{10}$$

where,

(i) $\sum_{k=1}^K v_k = 0$,

(ii) $\sum_{k=1}^K b_{kn} v_k = 0 \quad \forall n = 1, \ldots, N$,

(iii) $\lim\limits_{K\to\infty} \frac{1}{K} \sum_{k=1}^K v_k^2 = 0$.

Condition (iii) says that the average squared error of the pricing rule (10) goes to zero as K becomes large. So expected returns on average become closely approximated by (10). Notice that if the economy contains a risk-free asset then $\lambda_0 = R_f$.

Proof. (Back, page 115). □

Claim. A multibeta generalization of the CAPM with weaker assumptions as an economy with N priced sources of risk can always be described by an economy with only one priced source of risk (see Back, exercise #6.5, page 118).

Remark. The APT gives no guidance about candidates for the economy's multiple underlying risks. For example, Chen, Roll, and Ross choose macroeconomic factors; Fama and French choose empirically driven (firm-specific) factors that best fit cross-section of returns; Carhart added a fourth empirically driven momentum factor.

V. Hansen-Jagannathan Lower Bound on Risk Premia

Recall from the binomial model that,

$$S_{k,0} = E\left[\tilde{\phi}\tilde{S}_k\right] \Rightarrow E\left[\tilde{\phi}\tilde{R}_k\right] = 1.$$

Also $cov\left(\tilde{\phi}, \tilde{R}_k\right) + \bar{\phi}\bar{R}_k = 1$ by $cov\left(\tilde{x}\tilde{y}\right) = E\left[\tilde{x}\tilde{y}\right] - \bar{x}\bar{y}$. If $\exists$ a riskless asset with risk-free return R_f then $\bar{\phi}R_f = 1 \Rightarrow \bar{\phi} = \frac{1}{R_f}$.

Substituting $\Rightarrow cov\left(\tilde{\phi}, \tilde{R}_k\right) + \frac{1}{R_f}\bar{R}_k = 1 \Rightarrow R_f cov\left(\tilde{\phi}, \tilde{R}_k\right) + \bar{R}_k = R_f$. Thus,

$$\bar{R}_k - R_f = -R_f cov\left(\tilde{\phi}, \tilde{R}_k\right). \tag{11}$$

Let $\tilde{\phi} = SDF$ and $\bar{R}_k - R_f = -\frac{cov\left(\tilde{\phi}, \tilde{R}_k\right).}{\bar{\phi}} = -\frac{\rho_{k\tilde{\phi}}\sigma_k\sigma_{\tilde{\phi}}}{\bar{\phi}}$. Thus,

$$-\frac{\sigma_k\sigma_{\tilde{\phi}}}{\bar{\phi}} \le \frac{\rho_{k\tilde{\phi}}\sigma_k\sigma_{\tilde{\phi}}}{\bar{\phi}} \le \frac{\sigma_k\sigma_{\tilde{\phi}}}{\bar{\phi}},$$

as $\rho_{k\tilde{\phi}} \in [-1,1]$. Making substitutions,

$$\Leftrightarrow -\frac{\sigma_k \sigma_{\tilde{\phi}}}{\bar{\phi}} \leq \bar{R}_k - R_f \leq \frac{\sigma_k \sigma_{\tilde{\phi}}}{\bar{\phi}},$$

Rearranging terms,

$$\Leftrightarrow -\frac{\sigma_{\tilde{\phi}}}{\bar{\phi}} \leq \underbrace{\frac{\bar{R}_k - R_f}{\sigma_k}}_{Sharpe's\ ratio} \leq \frac{\sigma_{\tilde{\phi}}}{\bar{\phi}},$$

$$|Sharpe\ ratio| \leq \frac{\sigma_{\tilde{\phi}}}{\bar{\phi}}. \tag{12}$$

Remark. For the mean-variance efficient portfolio, the relation in equation (12) is a strict equality.

V.1 Mehra-Prescott Equity Risk Premium Puzzle. Let $\tilde{\phi} = \frac{\beta U'(C_1)}{U'(C_0)}$ where C is aggregate consumption.

(A1) $lnC_1 = lnC_0 + \varepsilon \Leftrightarrow C_1 = C_0 e^{\varepsilon}$ where $\varepsilon \sim N\left(\mu_\varepsilon, \sigma_\varepsilon^2\right)$ continuously compounded growth rate of consumption.

(A2) RA with power utility function $U\left(C\right) = \frac{1}{1-\theta} C^{1-\theta}$ where θ= coefficient of absolute risk aversion. Hence,

$$\tilde{\phi} = \beta \left(\frac{C_1}{C_0}\right)^{-\theta},$$

taking logs and exponentials gives,

$$= e^{ln\beta\left(\frac{C_1}{C_0}\right)^{-\theta}} = e^{ln\beta - \theta(lnC_1 - lnC_0)} = \beta e^{-\theta\varepsilon},$$

where $-\theta\varepsilon \sim N\left(-\theta\mu_\varepsilon, \theta^2\sigma_\varepsilon^2\right) \Rightarrow E\left[\tilde{\phi}\right] = \beta e^{-\theta\mu_\varepsilon + \frac{1}{2}\theta^2\sigma_\varepsilon^2}$ and $\sigma^2_{\tilde{\phi}} = E\left[\tilde{\phi}^2\right] - \bar{\phi}^2$ by definition of variance with $\tilde{\phi}^2 = \beta^2 e^{-2\theta\varepsilon}$.

Let $-2\theta\varepsilon \sim N\left(-2\theta\mu_\varepsilon, 4\theta^2\sigma_\varepsilon^2\right)$. Then, $E\left[\tilde{\phi}^2\right] = \beta^2 e^{-2\theta\mu_\varepsilon + 2\theta^2\sigma_\varepsilon^2}$ and $\bar{\phi}^2 = \beta^2 e^{-2\theta\mu_\varepsilon + \theta^2\sigma_\varepsilon^2}$. So, $E\left[\tilde{\phi}^2\right] = \bar{\phi}^2 e^{\theta^2\sigma_\varepsilon^2}$ and $\sigma^2_{\tilde{\phi}} = \bar{\phi}^2\left(e^{\theta^2\sigma_\varepsilon^2} - 1\right)$

$\Rightarrow \frac{\sigma_{\tilde{\phi}}}{\bar{\phi}} = \sqrt{e^{\theta^2\sigma_\varepsilon^2} - 1}$. Taking Taylor series expansion around 0 s.t. $e^x = e^0 + e^0 x + higher\ order\ terms = 1 + x + o\left(\cdot\right)$. For small $x \Rightarrow e^x \approx 1 + x$

$\Leftrightarrow \frac{\sigma_{\tilde{\phi}}}{\bar{\phi}} \approx \sqrt{1 + \theta^2\sigma_\varepsilon^2 - 1} = \theta\sigma_\varepsilon$. Thus,

$$|Sharpe's\ ratio| \leq \theta\sigma, \tag{13}$$

by H&J lower bound. For the S&P 500 $|Sharpe's\ ratio| = 0.49\ p.a.$ and $0.01 \leq \sigma \leq 0.04$. Then, 1) if $\sigma = 0.01 \Rightarrow \theta \geq 49$, 2) $\sigma = 0.04 \Rightarrow \theta \geq 12$. Too high! shouldn't be more than ~ 2.5!

References

[Black (1972) Journal of Business]
[Markowitz (1952) Journal of Finance]
[Roll (1977) Journal of Financial Economics]
[Reisman (1988) Econometrica]
[Hansen-Jagannathan (1991) Journal of Political Economy]