LECTURE NOTES 9 - PART B

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1. Production based models

1.1. **Production based model of expected stock returns in reduced form.** From the production side, firms can be seen as a portfolio of projects that adds value to the firm through its assets and growth options. Thus, the firm's value is the sum of the value of its assets in place and growth options.

(A.1) Infinite horizon and discrete time setting.
(A.2) At each date $j = 1, 2, \ldots$ a new project arises
(A.3) If project is accepted at date j and is still alive at date $t > j$ then it may be still alive at date $t+1$ with probability π, which is assumed to be independent of everything else.
(A.4) If it is still alive at $t+j$, the project's cash flow at time t per unit invested is equal to $C_j(t) = exp\left(\overline{C} - \frac{1}{2}\sigma_j^2 + \sigma_j \varepsilon_j(t)\right)$ for some (expected) value $\overline{C}$; and $\varepsilon_j(t)$ is normal, independent across j, t, and everything else.
(A.5) Size of each project (amount of irreversible investment or sunk cost) is equal to I. Thus, actual cash flow at time t is equal to $IC_j(t)$ and the expected cash flow if the project is still alive at time t is equal to $E\left[actual\ cash\ flow\,|alive\ \ at\ t\right] = Iexp\left(\overline{C}\right)$.
(A.6) The stochastic discount factor (SDF) of the underlying economy is $\frac{M_{t+1}}{M_t} = exp\left(-r(t) - \frac{1}{2}\sigma_M^2 + \sigma_M \upsilon(t+1)\right)$ where υ is normal and independent of everything. Recall that $\triangle logM = -r(t) - \frac{1}{2}\sigma_M^2 + \sigma_M \upsilon(t+1) = \left(-r(t) - \frac{1}{2}\sigma_M^2\right)\triangle t + \sigma_M \triangle B(t+1)$ so $\frac{dM}{M} = \sigma_M dB - rdt$ where B denotes Browniam motion.
(A.7) Riskless rate of return is modeled following Vasicek (1977): $r(t+1) - r(t) = (1-x)\left(\overline{r} - r(t)\right) + \sigma_M^2 \triangle t$.

Firm's value of its assets in place. The present value of $C_j(t)$ at date t assuming the project is still alive is,

$$E_t\left[\frac{M_{t+1}}{M_t} C_j(t+1)\right] = E_t\left[exp\left(-r(t) - \frac{1}{2}\sigma_M^2 + \sigma_M \triangle B_M\right) \times exp\left(\overline{C} - \frac{1}{2}\sigma_j^2 + \sigma_j \triangle B_j\right)\right],$$

where $\sigma_M \triangle B_M + \sigma_j \triangle B_j \sim N\left(0, \sigma_j^2 + \sigma_j^2 - 2\rho\sigma_M\sigma_j\right)$ with $\rho = cov\left(B_M, B_j\right)$. Solving the inner product in the expectation gives $exp\left(-r(t) - \frac{1}{2}\sigma_M^2 + \overline{C} - \frac{1}{2}\sigma_j^2 + \frac{1}{2}\sigma_M^2 + \frac{1}{2}\sigma_j + \rho\sigma_M\sigma_j\right) = exp\left(-r(t) + \overline{C} + \rho\sigma_M\sigma_j\right) = exp\left(\overline{C} - \beta_j\right) \times P(t, t+1)$, where $\beta_j = -\rho\sigma_M\sigma_j$ and $P(t, t+1)$ is the price at time t of the risk neutral (or certainty equivalent) cash flow at time $t+1$.

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Thus, the value of project j at time t is,

$$V_j(t) = \sum_{s=t+1}^{\infty} \pi^{s-t} Iexp\left(\overline{C} - \beta_j\right) P(t,s). \tag{1.1}$$

and the value of the firm's assets in place is,

$$\sum_{j=1}^{t} value\, of\, project\, j\, at\, date\, t\, if\, still\, alive = \sum_{j=1}^{t} V_j(t)\, 1_{\{j\}}(t),$$

where $1_{\{j\}}$ is an indicator function that equals 1 if project j was taken at date j and still alive at time t and 0 otherwise. Define $n(t) \equiv \#\, projects\, alive\, at\, t = \sum_{j=1}^{t} 1_{\{j\}}$ and $b(t) \equiv book\, value\, of\, firm\, at\, time\, t = In(t)$ ignoring depreciation. Then the value of the firm's assets in place can be re-written as,

$$\sum_{j=1}^{t} V_j(t)\, 1_{\{j\}}(t) = \sum_{s=t+1}^{\infty} \sum_{j=1}^{t} \pi^{s-t} Iexp\left(\overline{C} - \beta_j\right) P(t,s)\, 1_{\{j\}}(t) = \tag{1.2}$$

$$= \textstyle\sum_{s=t+1}^{\infty} \pi^{s-t} Iexp\left(\overline{C}\right) P(t,s) \sum_{j=1}^{t} exp\left(-\beta_j\right) 1_{\{j\}}(t)$$
$$= \textstyle\sum_{s=t+1}^{\infty} \pi^{s-t} Iexp\left(\overline{C}\right) P(t,s)\, exp\left(-\beta(t)\right) n(t),$$

where $\beta(t) \equiv \frac{\sum_{j=1}^{t} exp\left(-\beta_j 1_{\{j\}}(t)\right)}{n(t)}$ is the average $exp\left(-\beta_j\right)$ over all projects alive. Rearranging terms,

$$\sum_{j=1}^{t} V_j(t)\, 1_{\{j\}}(t) = Iexp\left(\overline{C} - \beta(t)\right) n(t) \sum_{s=t+1}^{\infty} \pi^{s-t} P(t,s). \tag{1.3}$$

If we define $D(r(t)) = \sum_{s=t+1}^{\infty} \pi^{s-t} P(t,s)$ as the price of a default-free bond that pays n^{s-t} at each date $s > t$ then

$$Value\, of\, assets\, in\, place = b(t)\, exp\left(\overline{C} - \beta(t)\right) D(r(t)). \tag{1.4}$$

Firm's value of its growth options. From the fundamental asset pricing equation,

$$Value\, of\, growth\, options = \sum_{s=t+1}^{\infty} E_t\left[\frac{M(s)}{M(t)} Max\left(V_s(s) - I, 0\right)\right]. \tag{1.5}$$

Recall that $V_s(s) = Iexp\left(\overline{C} - \beta_s\right) D(s)$, consequently given the optimal value of a perpetual American call option if not excercised before time t i.e., $J^*(r(t))$ we can re-express the value in equilibrium of the firm's growth options as

$$Value\, of\, growth\, options = Iexp\left(\overline{C}\right) J^*(r(t)). \tag{1.6}$$

Expected returns. The price of the firm at time t can be written as,

$$p(t) = b(t)\, exp\left(\overline{C} - \beta(t)\right) D(r(t)) + Iexp\left(\overline{C}\right) J^*(r(t)), \tag{1.7}$$

Note that from equation 1.7, $\beta(t)$ can be expressed as a function of observables i.e., stock prices, economic growth rates, riskless bond prices, and implied option prices. The next step is to find the expected return equation in equilibrum

$$E_t\left[1 + r(t+1)\right] =$$

$$\frac{\pi b\left(t\right) exp\left(\overline{C}-\beta\left(t\right)\right) E_t\left[D\left(r\left(t+1\right)\right)\right]+\pi b\left(t\right) exp\left(\overline{C}\right)+I exp\left(\overline{C}\right) E_t\left[J^*\left(r\left(t+1\right)\right)\right]}{b\left(t\right) exp\left(\overline{C}-\beta\left(t\right)\right) D\left(r\left(t\right)\right)+I exp\left(\overline{C}\right) J\left(r\left(t\right)\right)},$$

and

$$\Rightarrow E_t\left[1+r\left(t+1\right)\right]=\underbrace{\frac{\pi E_t\left[D\left(r\left(t+1\right)\right)\right]}{D\left(r\left(t+1\right)\right)}}_{same\ for\ every\ firm}+$$

(1.8)

$$+\pi \qquad \underbrace{exp\left(\overline{C}\right)\underbrace{\frac{b\left(t\right)}{P\left(t\right)}}_{BEME}}_{close\ to\ earnings\ per\ share\ (EPS)} \qquad \underbrace{+I exp\left(\overline{C}\right)\left\{E_t\left[J^*\left(r\left(t+1\right)\right)\right]-\frac{\pi E_t\left[D\left(r\left(t+1\right)\right)\right]}{D\left(r\left(t+1\right)\right)}J^*\left(r\left(t\right)\right)\right\}}_{same\ for\ every\ firm}\underbrace{\frac{1}{P\left(t\right)}}_{ME}$$

where $BEME$ denotes book to market and ME size.

References

[Berk, Green, and Naik (1999) Journal of Finance]
[Vasicek (1977) Journal of Financial Economics]