LECTURE NOTES 8

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1. Term Structure Models

1.1. **Basic formulation.** Let $y(t,T)$ be the bond equivalent yield (bey) at time t of a discount bond with maturity T. Then,

$$P(t,T) = e^{-y(t,T)(T-t)},$$

is the price of the discount bond in continuous time. Under risk-neutral probability,

$$\frac{P(t,T)}{R(t)} = Martingale,$$

where $R(t) = e^{\int_0^t r(s)ds}$; and r is the short-term risk-free interest rate. Thus,

$$\frac{P(t,T)}{R(t)} = E_t^R\left[\frac{P(T,T)}{R(T)}\right],$$

where $P(t,T)$ is the price at time t of a default-free bond that pays 1 dollar at maturity T with $E_t^R\left[R(T)^{-1}\right]$; and we write R instead of Q. Thus, the fundamental pricing formula for default-free bonds is,

$$P(t,T) = E_t^R\left[\frac{R(t)}{R(T)}\right] = E_t^R\left[e^{-\int_t^T r(s)ds}\right]. \tag{1.1}$$

The goal is to model the short-term interest rate $r(s)$ under the risk-neutral measure and then price the rest of the bonds along the yield curve assuming no-arbitrage.

1.2. **Vasicek (1977) model.** The short-term interest rate is modeled as the following Ornstein-Uhlenbeck (i.e., mean-reverting) process,

$$dr = \kappa(\theta - r)\,dt + \sigma dB, \tag{1.2}$$

where B is a Brownian motion under the risk neutral measure; θ is the long-run mean; k is the speed of mean-reversion; and σ is the volatility. All parameters are constant by assumption. Let $\kappa = 0$ (i.e., no mean-reversion). Thus,

$$dr = \sigma dB.$$

For $s > t$,

$$r(s) = r(t) + \sigma[B(t) - B(s)] \sim N\left(r(t), \sigma^2(s-t)\right).$$

We want to calculate $E_t\left[e^{-\int_t^T r(s)ds}\right]$. Thus,

$$r(s) = r(t) + \sigma\int_t^s dB(u),$$

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$$\int_t^T r(s)\,ds = \int_t^T \left[r(t) + \sigma \int_t^s dB(u) \right] ds$$

$$= (T-t)\,r(t) + \sigma \int_t^T \int_t^s dB(u)\,ds$$

$$= (T-t)\,r(t) + \sigma \int_t^s \int_u^T ds dB(u)\,, \quad by\ Fubini's\ Theorem$$

$$= (T-t)\,r(t) + \sigma \int_t^T (T-u)\,dB(u) \sim N\left((T-t)\,r(t)\,, \sigma^2 \int_t^T (T-u)^2\,du \right).$$

Note. Fubini's theorem is a mathematical result that establishes the required conditions to change the order of integration.

Notice that the extra return over the riskless interest rate per unit of volatility is independent of the underlying short-term rate to avoid arbitrage (as in the BSM approach), consequently the decomposition is unique. For the mean-reverting process we retain independence across time assuming a Markov stationary process,

$$r(s) = r(t) + \int_t^s \kappa\,(\theta - r(u))\,du + \sigma dB(u)\,. \tag{1.3}$$

One problem with this model is Normality i.e., allows the short interest rate to become negative.

1.3. **CIR (1985) model.** In order to satisfy the lower zero-bound constraint in interest rates, the short-term interest rate is modeled as the following (O-U) mean-reverting process,

$$dr = \kappa\,(\theta - r)\,dt + \sigma\sqrt{r}dB, \tag{1.4}$$

where B is a Brownian motion under the risk neutral measure; θ is the long-run mean; k is the speed of mean-reversion; and σ is the volatility. All parameters are constant by assumption like in Vasicek's model.

Conditional on $r(t)$, for $s > t$ $r(s)$ has non-central chi-squared distribution. When r is too low, that is ~ 0, the drift dominates the diffussion term and r can't become negative as in Vasicek's model.

Find $E_t^R\left[e^{-\int_t^T r(s)ds}\right]$. Guess a solution $P(t,T) = e^{-a(T-t)-b(T-t)r(t)}$, where a, b are deterministic functions of time to maturity. Recall that the reason for the negative sign is that $P(t,T) = e^{-y(t,T)(T-t)}$ implies $y(t,T) = -\frac{1}{T-t}logP(t,T)$. Thus,

$$y(t,T) = \frac{a(T-t)}{T-t} + \frac{b(T-t)}{T-t}r(t)\,.$$

The goal now is to find a and b. Start with the guess,

$$P(t,T) = e^{X(t)},$$

where $X(t) = -a(T-t) - b(T-t)\,r(t)$. By Ito's lemma,

$$\frac{dP}{P} = dX + \frac{1}{2}(dX)^2\,,$$

where $dX = a'(T-t)\,dt + b'(T-t)\,r(t)\,dt - b(T-t)\,dr$; and $(dX)^2 = [b(T-t)]^2(dr)^2 = [b(T-t)]^2\sigma^2 r dt$. Substituting these terms in the equation for price dynamics gives,

$$\frac{dP}{P} = \left(a' + b'r - b\kappa(\theta - r) + \frac{1}{2}b^2\sigma^2 r\right)dt - b\sigma\sqrt{r}dB.$$

By the fundamental theorem of asset pricing the PDE for the expected return r is,

$$a' + b'r - b\kappa(\theta - r) + \frac{1}{2}b^2\sigma^2 r,$$

with boundary conditions,

$$a(0) = 0,$$
$$b(0) = 0.$$

Because $P(t,T) = e^{-a(T-t)-b(T-t)r(t)} = 1 \Rightarrow -a(T-t) - b(T-t)\,r(t) = 0$,

$$\Rightarrow a' - b\kappa\theta + \left(b' + b\kappa + \frac{1}{2}\sigma^2 b^2 - 1\right)r = 0, \quad \forall r$$

$$\Rightarrow \begin{cases} a' = b\kappa\theta \\ b' + \kappa b + \frac{1}{2}\sigma^2 b^2 = 1 \end{cases},$$

where $a(u) = a(0) + \int_0^u a'(s)\,ds = 0 + \int_0^u \kappa\theta b(s)\,ds = 0 + \int_0^u \kappa\theta b(s)\,ds$, and $b(u)$ can be obtained solving the Ricatti equation.

The empirical evidence for one-factor models using market prices of bonds show that these models do a poor job in explaining the yield curve. The problem is the stationarity assumption that does not hold over long-periods of time. This empirical fact is compatible with regime-switching models that allow for a jump in the drift and/or volatility of the short-term interest rate. Thus, one can adjust the one-factor models to allow a jump in the parameter θ; or alternatively to be time-varying as in Hull and White (1993) fitting/calibrating the observed yield curve.

1.4. **Longstaff-Schwartz (1992) model.** Given the empirical limitations of one-factor models to fit the observed yield curve, the short-term interest rate has been modeled as a two-factor model,

$$r(t) = X_1(t) + X_2(t),$$

where,

$$dX_1 = \kappa_1(\theta_1 - X_1)\,dt + \sigma_1\sqrt{X_1}dB_1,$$
$$dX_2 = \kappa_2(\theta_2 - X_2)\,dt + \sigma_2\sqrt{X_2}dB_2,$$

with B_1, B_2 are two independent Brownian motions under risk-neutral probability i.e., $BB^T = \begin{pmatrix} \sigma_1^2 X_1 & 0 \\ 0 & \sigma_2^2 X_2 \end{pmatrix}$.

Then,

$$P(t,T) = E^R\left[e^{-\int_t^T r(s)ds}\right] = E^R\left[e^{-\int_t^T X_1(s)ds - \int_t^T X_2(s)ds}\right]$$
$$= E_t^R\left[e^{-\int_t^T X_1(s)ds}e^{-\int_t^T X_2(s)ds}\right] = E_t^R\left[e^{-\int_t^T X_1(s)ds}\right]E_t^R\left[e^{-\int_t^T X_2(s)ds}\right]$$
$$= P_1(t,T) \times P_2(t,T),$$

where $\forall i \quad P_i(t,T)$ is the bond pricing formula using the CIR model. Thus,

$$P_i(t,T) = e^{-a_i(T-t)-b_i(T-t)X_i(t)},$$

and,

$$(dr)^2 = (dX_1 + dX_2)^2 = (dX_1)^2 + (dX_2)^2 + 2\,(dX_1)\,(dX_2) = \sigma_1^2 X_1 dt + \sigma_2^2 X_2 dt + 2\times 0 \quad \textit{by independence.}$$

Define,

$$V(t) = \sigma_1^2 X_1 + \sigma_2^2 X_2,$$

i.e., both r and V are affine functions of two factors. Then,

$$\begin{pmatrix} V \\ r \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} V \\ r \end{pmatrix}.$$

Substituting in the pricing formula,

$$P(t,T) = e^{-a_1(T-t) - a_2(T-t) - \begin{pmatrix} b_1(T-t) \\ b_2(T-t) \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} V \\ r \end{pmatrix}}, \tag{1.5}$$

$$dr = \left[\kappa_1\theta_1 + \kappa_2\theta_2 - \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix}^T \begin{pmatrix} \sigma_1^2 & \sigma_2^2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} V \\ r \end{pmatrix}\right] dt + V^{\frac{1}{2}} dB, \tag{1.6}$$

where $dV = \cdots dB^T$(Brownian motion are now correlated).

This model is capable of obtaining a very good fit to a variety of actual/observed yield curves, and to the first two moments of the conditional distribution of the short-term rate. The problem is that the observed correlation between the first two moments is too high (90%-99%) implying that the model is very close to the one-factor models of Vasicek and CIR.

1.5. **Affine models.** Affine models are driven by the joint requirement of calibration (recovering prices from the observed yield curve) and ease of computation. The alternative to this approach, is to drop the Markov property as in the third-generation models pioneered by Heath, Jarrow, and Morton (1992). The problem with this model is that relies in forward-induction and Monte-Carlo techniques given that the short-term interest rate is now path-dependent. This leads to a costly trade-off between good calibration and computation tractability.

Let X be a $n \times 1$ state vector, then r is an affine function of X such that,

$$dX = \alpha(X)\,dt + \beta(X)\,dB, \tag{1.7}$$

is Markov; and $\alpha, \beta\beta^T$ are affine functions of X.

1.6. **Factor interpretation of Affine DTSMs.** Using principal component (PC) analysis and observed zero-coupon yields with different maturities, the empirical literature on dynamic term structure models (DTSMs) have found that a low dimensional Markov state vector including three latent risk factors capturing the level, slope, and curvature of the yield curve does a good job in tracking the dynamics of the yield curve retaining ease of computation. There is also a large descriptive evidence that macroeconomic variables are highly correlated with these three latent state variables.

We decompose the response of the bond yield with maturity T to innovations in a set of risk factors that affect expected future changes in the short-term interest rate ES and the term premium TP,

$$R_t^T = \frac{1}{T}\sum_{s=0}^{T-1} E_t\left[r_{t+s}\right] + \frac{1}{T}\sum_{s=0}^{T-1} p_t^s \equiv ES_t\left(T\right) + TP_t\left(T\right). \tag{1.8}$$

In the spirit of Brennan and Schwartz (1982), the term structure is driven by two state variables: the short-term interest rate and a consol yield. Evans and Marshall (2001) found that shocks in aggregate demand are highly correlated with the first PC or "level factor" of the yield curve, with a half-life (i.e., persistence) of 2.65 years. Shifts in the stance of monetary policy affect the "slope factor" of the yield curve as monetary policy is more effective in the short-end of the yield curve, with a half-life of 1 year. The third factor is the "curvature factor" that seems to reflect shocks to flight to quality (i.e., investors' sentiment not related to fundamentals) with a half-life of only 2.2 months.

The level factor affects the short-end of the yield curve through expected future changes in short interest rates ES mainly. As maturity increases the effect through ES decreases and the effect through the TP increases. For example, for a 10 year T-bond the level factor exerts influence in the yield curve equally through ES and TP. The slope factor affects yields virtually entirely through ES having near-zero effect on TP. Finally the curvature factor is a pure risk-premium phenomenon with a near-zero ES effect.

The second generation of empirical macroeconomic-based DTSMs seek to integrate the dynamics of the yield curve with those of the business cycle. Ang and Piazzesi (2003) estimate a five-factor affine model with three latent factors and two macroeconomic observed macroeconomic variables (given Taylor's rule): level of economic activity, and inflation. Other strand of this literature introduces multiple regimes into the affine specification with latent risk factors matching the complex dynamics of the actual real business cycle.

Notice that this link is consistent with the factor representation of systematic risk in the ICAPM when the time-varying opportunity set is assumed to be time-varying.

References

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