LECTURE NOTES 7

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1. Dynamic Programming (A Brief Review)

1.1. Discrete time problem.

Let time $t \in [0,T]$ with $u(c_0), \delta u(c_1), \ldots, \delta^{T-1}u(c_{T-1}), \delta^T u(c_T) = B(W_T)$(i.e., bequest).

(A.1) No labor income.
(A.2) At date $t = 0$, W_0 is given as an initial endowment.
(A.3) W_t = wealth before consumption at date t.
(A.4) Invest $W_t - c_t$ at date t.
(A.5) Choose π_t, vector of portfolio weights.

Wealth at time $t+1$ is equal to $(W_t - c_t)\, \pi_t^T R_t$ where $R_t = \begin{pmatrix} R_{t+1,1} \\ \vdots \\ R_{t+1,n} \end{pmatrix}$ is the vector of returns over $[t, t+1]$.

The dynamic behavioral problem is stated,

$$\underset{c}{Max}\; u(c_0) + E\left[\sum_{t=1}^{T-1} \delta^t u(c_t)\right] + \delta^T E\left[B(W_T)\right].$$

We solve this problem using backward induction,

At date T.

Given W_T choose c_T to $Max\;\; u(c_T) + B(W_T - c_T)$ and define $J(W_T, T) = max.value$.

At date $T-1$.

Given W_{T-1} choose c_{T-1} to $Max\; u(c_{T-1}) + \delta E[J(W_T, T)]\,,\; s.t.\; W_T = (W_{T-1} - c_{T-1})\, \pi_{T-1} \tilde{R}_T$ and define $J(W_{T-1}, T-1) = max.value$.

At date $T-2$.

Given W_{T-2} choose c_{T-2} to $Max\; u(c_{T-2}) + \delta E[J(W_{T-1}, T-1)]\,,\; s.t.\; W_{T-1} = (W_{T-2} - c_{T-2})\, \pi_{T-2} \tilde{R}_{T-1}$ and define $J(W_{T-2}, T-2) = max.value$.

And so on...

1.2. Bellman's principle of optimality in continuous time.

From the previous analysis we know that

$J(W,T) = \underset{\{portfolio, consumption\}}{Max} E\left[u(c(t)) + \delta E[J(W', t+1)]\right]$, where W' is the level of wealth at time $t+1$. Assume consumption only at time T and define $\hat{u}(c) = \delta^T u(c)$, then drop $\hat{}$ so that the problem becomes $MaxE[u(W(T))]$. Therefore, by principle of optimality,

$$J(W,t) = MaxE[J(W', t+1)] \Rightarrow 0 = MaxE[J(W', t+1) - J(W,t)],$$

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Assume continuous trading and consumption, then by Bellman's principle,

$$0 = Max\left[drift\ of\ J\right].$$

Assume i.i.d. returns with value function $J(W,t)$ then by Ito's lemma,

$$dJ = \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial W}dW + \frac{1}{2}\frac{\partial^2 J}{dW^2}(dW)^2. \tag{1.1}$$

Claim. The Bellman equation satifies the transversality condition $u(W(T)) = J(W(T),T)$ for any plausible policy and Bellman's principle for the optimal policy.

Proof. It must be $u(W(T)) = J(W(0),0) + \int_0^T dJ = J(W(0),0) + \int_0^T drift\ of\ J + \int_0^T something\ dB$

$$\Rightarrow E\left[u(W(T))\right] = J(W(0),0) + E\left[\int_0^T drift\ of\ J\right] + \underbrace{E\left[\int_0^T something\ dB\right]}_{=0}.$$

Then it must be that $E\left[\int_0^T drift\ of\ J\right] = 0$ if the policy is optimal (Dynamic consistency). □

Example. BSM model with $q = 0$.

$$\frac{dW}{W} = rdt + \pi(\mu - r)dt + \pi\sigma dB,$$

$$drift\ J = \frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial W}\left[r + \pi(\mu - r)\right] + \frac{1}{2}\frac{\partial^2 J}{dW^2}W^2\pi^2\sigma^2.$$

And the Bellman equation is $0 = \underset{\pi}{Max}\left\{\frac{\partial J}{\partial t}dt + \frac{\partial J}{\partial W}\left[r + \pi(\mu - r)\right] + \frac{1}{2}\frac{\partial^2 J}{dW^2}W^2\pi^2\sigma^2\right\}$, with transversality condition $J(W,T) = u(W)$.

2. Intertemporal Asset Pricing

2.1. **Breeden's consumption-CAPM (CCAPM).** Let u^h = utility function of the hth investor. Recall that under fundamental asset pricing for each asset,

$$(risk\ premium)\,dt = -\left(\frac{dS_k}{S_k}\right)\left(\frac{dM^h}{M^h}\right),$$

where $M^h(t) = f\left(C^h(t),t\right)$ with f denoting marginal utility. By Ito's lemma,

$$\begin{aligned} dM^h &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial C^h}dC^h + \frac{1}{2}\frac{\partial^2 f}{\partial\left(C^h\right)^2}\left(dC^h\right)^2 \\ &= something\ dt + \frac{\partial f}{\partial C^h}dC^h \\ &= something\ dt + \frac{\partial^2 u\left(C^h(t),t\right)}{\partial\left(C^h\right)^2}dC^h. \end{aligned}$$

Write $u' = \frac{\partial u}{\partial C}$ and $u'' = \frac{\partial^2 u}{\partial C^2}$ then,

$$\frac{dM^h}{M^h} = \frac{u''\left(C^h(t),t\right)}{u'\left(C^h(t),t\right)}dC^h.$$

Assume $h=$ (RA) and multiply and divide by C^h that is,

$$\frac{dM^h}{M^h} = \frac{C^h u''\left(C^h(t),t\right)}{u'\left(C^h(t),t\right)}\frac{dC^h}{C^h}$$

$$= -\left(Arrow - Pratt\ coefficient\ of\ RRA\right) \times \frac{dC^h}{C^h}$$

$$\Rightarrow (risk\ premium)\,dt = CRRA \times \left(\frac{dS_k}{S_k}\right)\left(\frac{dC^h}{C^h}\right). \tag{2.1}$$

Without the (RA) assumption we still have,

$$\Rightarrow (risk\ premium)\,dt = -\frac{C^h u''\left(C^h(t),t\right)}{u'\left(C^h(t),t\right)} \times \left(\frac{dS_k}{S_k}\right)\left(\frac{dC^h}{C^h}\right)$$

$$\Leftrightarrow \frac{u'\left(C^h(t),t\right)}{u''\left(C^h(t),t\right)}(risk\ premium)\,dt = -\left(\frac{dS_k}{S_k}\right)\left(dC^h\right),$$

so that after adding across h investors we get,

$$-\sum_h \frac{u'\left(C^h(t),t\right)}{u''\left(C^h(t),t\right)}(risk\ premium)\,dt = \left(\frac{dS_k}{S_k}\right)\left(dC^{aggregate}\right)$$

$$(risk\ premium)\,dt = \underbrace{\left(-\sum_h \frac{u'\left(C^h(t),t\right)}{u''\left(C^h(t),t\right)}\right)^{-1}}_{reciprocal\ of\ sum\ of\ inverse\ CARA}\left(\frac{dS_k}{S_k}\right)\left(dC^{aggregate}\right). \tag{2.2}$$

2.2. **Merton's intertemporal-CAPM (ICAPM).** Let X =vector of $n-$factors, with k assets, and a vector of $n+k$ Brownian motions B. Assuming X follows a Markov process then we have,

$$\underbrace{dX(t)}_{n\times 1} = a\left(\underbrace{X(t)}_{n\times 1}\right)dt + \underbrace{b\left(X(t)\right)}_{n\times(n+k)}\underbrace{dB(t)}_{(n+k)\times 1}.$$

With dynamics for the risky assets following a vector of geometric brownian motions,

$$\begin{pmatrix} \frac{dS_1}{S_1} \\ \vdots \\ \frac{dS_k}{S_k} \end{pmatrix} = \underbrace{\frac{dS(t)}{S(t)}}_{k\times 1} = \underbrace{\mu\left(X(t)\right)}_{k\times 1}dt + \underbrace{\sigma\left(X(t)\right)}_{k\times(n+k)}\underbrace{dB(t)}_{(n+k)\times 1},$$

and the short interest rate,

$$dr(t) = r\left(X(t)\right)dt + \kappa\left(X(t)\right)dB(t).$$

The investors' dynamic behavioral problem can be stated as,

$$\underset{\{C,\pi_i\}}{Max}\left\{E\int_0^T u\left(t,C(t)\right)dt\right\},$$

$$s.t.\ dW = r\left(1-\sum_{i=1}^{k}\pi_i\right)Wdt+\sum_{i=1}^{k}\pi_iW\left(\underbrace{\mu_idt+\sum_{j=1}^{n+k}\sigma_{i,j}dB_j}_{\frac{dS_i}{S_i}}-rdt\right)-Cdt.$$

That is, the investor maximizes utility by trading and consuming through time.

Define $J(W,X,t) = Max\left\{E\int_t^T u(s,C(s))\,ds\right\}$, then the corresponding Bellman equation is,

$$\underset{\{C,\pi\}}{Max}\left\{E[dJ]+u(t,C(t))\right\}=0, \tag{2.3}$$

with Dynkin operator (by Ito's lemma),

$$dJ=\frac{\partial J}{\partial W}dW+\sum_{i=1}^{n}\frac{\partial J}{\partial X_i}dX_i+\frac{\partial J}{\partial t}dt+\frac{1}{2}\frac{\partial^2 J}{\partial W^2}(dW)^2+\sum_{i=1}^{n}\frac{\partial^2 J}{\partial W\partial X_i}(dW)(dX_i)+$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 J}{\partial X_i\partial X_j}(dX_i)(dX_j). \tag{2.4}$$

Notice that maximizing over C is equivalent to maximizing over $u(C,t)-\frac{\partial J}{\partial W}$. Assuming an interior solution exists the F.O.N.C. w.r.t. C is,

$$-\frac{\partial J}{\partial W}+\frac{\partial u(C,t)}{\partial C}=0$$

$$\Leftrightarrow \frac{\partial u(C,t)}{\partial C}=\frac{\partial J}{\partial W}\quad (envelope\ theorem\ condition),$$

$$\Rightarrow \frac{\partial J}{\partial W}=constant\times SDF. \tag{2.5}$$

And the F.O.N.C. w.r.t. π is,

$$(risk\ premium)\,dt=-\left(\frac{dS_k}{S_k}\right)\left(\frac{d\frac{\partial J}{\partial W}}{\frac{\partial J}{\partial W}}\right), \tag{2.6}$$

where,

$$d\frac{\partial J}{\partial W}(W(t),X(t),t)=\frac{\partial^2 J}{\partial W^2}dW+\sum_{i=1}^{n}\frac{\partial^2 J}{\partial W\partial X_i}dX_i+\frac{\partial^2 J}{\partial W\partial t}dt+\frac{1}{2}\frac{\partial^3 J}{\partial W^3}(dW)^2+$$

$$+something\ dt.$$

Thus,

$$\left(\frac{dS_k}{S_k}\right)\left(\frac{d\frac{\partial J}{\partial W}}{\frac{\partial J}{\partial W}}\right)=\frac{\partial^2 J}{\partial W^2}dW+\sum_{i=1}^{n}\frac{\partial^2 J}{\partial W\partial X_i}dX_i+\mathcal{O}(t)$$

$$\Rightarrow (risk\ premium)\,dt=-\frac{\frac{\partial^2 J}{\partial W^2}}{\frac{\partial J}{\partial W}}\left(\frac{dS_k}{S_k}\right)(dW)-\sum_{i=1}^{n}\frac{\frac{\partial^2 J}{\partial W\partial X_i}}{\frac{\partial J}{\partial W}}\left(\frac{dS_k}{S_k}\right)(dX_i).$$

This holds for each investor h thus,

$$\sum_h\left(\frac{\frac{\partial J}{\partial W}}{\frac{\partial^2 J}{\partial W^2}}\right)(risk\ premium)\,dt=-\left(\frac{dS_k}{S_k}\right)\left(dW^{aggregate}\right)-$$

$$-\sum_h \left(\frac{\frac{\partial J}{\partial W}}{\frac{\partial^2 J}{\partial W^2}}\right) \sum_{i=1}^{n} \frac{\frac{\partial^2 J}{\partial W \partial X_i}}{\frac{\partial J}{\partial W}} \left(\frac{dS_k}{S_k}\right)(dX_i)$$

$$\Rightarrow (risk\ premium)\, dt = -\underbrace{\left(\sum_h \underbrace{\frac{1}{CARA^h}}_{in\ terms\ of\ J(W)}\right)^{-1}}_{\theta^{aggregate}(t)} \left(\frac{dS_k}{S_k}\right)\left(dW^{aggregate}\right) -$$

$$-\underbrace{\frac{\sum_h \left(\sum_{i=1}^{n} \frac{\frac{\partial^2 J^h}{\partial W \partial X_i}}{\frac{\partial^2 J^h}{\partial W^2}}\right)}{\sum_h \frac{1}{CARA^h}}}_{\lambda_j(\theta)} \left(\frac{dS_k}{S_k}\right)(dX_i)$$

(2.7)

$$\Rightarrow (risk\ premium)\, dt = -\theta^{aggregate}(t)\left(\frac{dS_k}{S_k}\right)\left(dW^{aggregate}\right) + \lambda_j \left(\frac{dS_k}{S_k}\right)(dX_i).$$

Note. We can differentiate three hedging funds: 1) $1-\pi$ invested in the riskless asset; 2) part of π is long in the MVE tangency portfolio; and 3) the rest is long/short in a portfolio with maximal correlation with the vector of state variables X depending on the sign of λ_j.

Intuition: Innovations in the vector of state variables X have an impact in the time-varying investment opportunity set, so investors need to hedge against changes in the investment opportunity set. The indeterminate sign in the second term in (7) is because it depends on innovations in X being bad or good news w.r.t. marginal utility.

Claim. For empirical purposes this means that the risk premium across test assets sould be some affine function of the market portfolio plus portfolios that are shown to have maximum correlation with innovations in the state variables.

Proof. It follows from previous analysis. □

2.3. **An Approximate CAPM.** (A1) The joint distribution of returns R_{t+1} and nonportfolio income for investor h i.e., $Y_{h,t+1}$ is independent of date-t information, for each t and h.

Or

(A1') Each investor has myopic preferences (i.e., log utility function) and zero endowments $Y_{h,t+1}$.

In either case, both assumptions imply that $J_{hwx_j=0}$ $\forall h, j$. This weak condition allows the derivation of the conditional CAPM as an approximation to the ICAPM.

References

[Merton (1973) Econometrica]
[Breeden (1979) Journal of Financial Economics]