LECTURE NOTES 6

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1. Conditional CAPM

Claim. We say that the CAPM holds if for each asset k and each state of the world s the fundamental asset pricing formula (i.e., the conditional CAPM),

$$\underbrace{risk\ premium \cdot dt}_{(fluctuating)} = risk\ premium\ W_m \times \frac{\left(\frac{dS}{S}\right)\left(\frac{dW_m}{W_m}\right)}{\left(\frac{dW_m}{W_m}\right)^2},$$

holds, and the maximal Sharpe's ratio is state-dependent.

Proof. Let S be the price of a non-dividend paying asset.

(A1) $\frac{dS_k}{S_k} = \mu_k dt + \sigma_k dB_k$.
(A2) $\frac{dW_m}{W_m} = \mu_m dt + \theta_m dB_m$.
(A3) $B_k + B_m$ have correlation ρ.
(A4) The drift, difussion, risk-free rate r, and market price of risk are all constant.
(A5) Fix time from 0 to 1.

Then in each state s and for each asset k it must be that,

$$\mu_k - r = (\mu_m - r) \times \frac{\sigma_k \theta \rho}{\theta^2},$$

by the fundamental asset pricing formula. Also,

$$S_k(1) = S_k(0)\, e^{\mu_k - \frac{1}{2}\sigma_k^2 + \sigma B_k(1)},$$

$$W_m(1) = W_m(0)\, e^{\mu_m + \theta^2 + \theta B_m(1)}.$$

Note that $E[S_k(1)] = S_k(0)\, e^{\mu_k}$ and $E[W_m(1)] = W_m(0)\, e^{\mu_m}$ so that $\mu_k = logE\left[\frac{S_k(1)}{S_k(0)}\right]$, $\mu_m = logE\left[\frac{W_m(1)}{W_m(0)}\right]$, $var\ log\frac{S_k(1)}{S_k(0)} = \sigma_k^2$, $var\ log\frac{W_m(1)}{W_m(0)} = \theta^2$, and $cov\left(log\frac{S_k(1)}{S_k(0)}, log\frac{W_m(1)}{W_m(0)}\right) = \sigma\theta\rho$. If the conditional CAPM holds, then in each state s it must be,

$$logE\left[\frac{S_k(1)}{S_k(0)}\right] - r = \underbrace{\left(logE\left[\frac{W_m(1)}{W_m(0)}\right] - r\right)}_{continuously\ compounded} \times \frac{cov\left(log\frac{S_k(1)}{S_k(0)}, log\frac{W_m(1)}{W_m(0)}\right)}{var\ log\frac{W_m(1)}{W_m(0)}}.$$

□

Note. For empirical work use continuously compound returns to estimate the betas and then run the second-pass regression using simple log returns.

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1.1. **From the pricing kernel (SDF) to a conditional factor model.** We assume that the pricing kernel or SDF is some affine function of a vector of N risk factors f_t,

$$M(t) = \phi^0_{t-1} + \phi^T_{f,t-1} f_t, \tag{1}$$

where $\tilde{\phi}^T_{t-1} = \left(\phi^0_{t-1}, \phi^T_{f,t-1}\right) \in \mathcal{F}_{t-1}$; $\mathcal{F}_{t-1}$ is the filtration or information set up to $t-1$; and $\tilde{f}^T_t = \left(1, f^T_t\right)$. The first conditional moment restriction is,

$$R^f_{t-1} E\left[M(t) \left|\mathcal{F}_{t-1}\right.\right] = 1 = R^f_{t-1} E\left[\tilde{f}_t \left|\mathcal{F}_{t-1}\right.\right]^T \tilde{\phi}_{t-1}, \tag{2}$$

and by fundamental asset pricing, the second moment restriction for each k risky asset is,

$$E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right] - r^f_{t-1} = -R^f_{t-1} cov\left(r_{k,t}, M(t)\left|\mathcal{F}_{t-1}\right.\right), \quad \forall r_{k,t} \in \mathbb{R}_{k,t}. \tag{3}$$

Substituting (2) into (3) gives,

$$E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right] - r^f_{t-1} = -\frac{cov\left(r_{k,t}, M(t)\left|\mathcal{F}_{t-1}\right.\right)}{E\left[M(t)\left|\mathcal{F}_{t-1}\right.\right]}, \quad \forall r_{k,t} \in \mathbb{R}_{k,t}.$$

Substituting (1) into this last expression and using properties of covariances gives,

$$E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right] - r^f_{t-1} = -\frac{\phi^f_{t-1} cov\left(r_{k,t}, f^T_t\left|\mathcal{F}_{t-1}\right.\right)}{E\left[M(t)\left|\mathcal{F}_{t-1}\right.\right]}, \quad \forall r_{k,t} \in \mathbb{R}_{k,t},$$

which by using (2) again can be re-written as,

$$E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right] - r^f_{t-1} = -\frac{\phi^f_{t-1} cov\left(r_{k,t}, f^T_t\left|\mathcal{F}_{t-1}\right.\right)}{E\left[\tilde{f}_t\left|\mathcal{F}_{t-1}\right.\right]^T \tilde{\phi}_{t-1}}, \quad \forall r_{k,t} \in \mathbb{R}_{k,t},$$

$$\Leftrightarrow E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right] - r^f_{t-1} = -\frac{\phi^f_{t-1} cov\left(r_{k,t}, f^T_t\left|\mathcal{F}_{t-1}\right.\right)}{E\left[\tilde{f}_t\left|\mathcal{F}_{t-1}\right.\right]^T \tilde{\phi}_{t-1}} \times \frac{cov\left(f_t, f^T_t\left|\mathcal{F}_{t-1}\right.\right)}{cov\left(f_t, f^T_t\left|\mathcal{F}_{t-1}\right.\right)}, \forall r_{k,t} \in \mathbb{R}_{k,t},$$

and finally substituting terms backwards and rearranging gives,

$$\beta^T_{k,t-1} \times \frac{-\phi^f_{t-1} cov\left(f_t, f^T_t\left|\mathcal{F}_{t-1}\right.\right)}{E\left[\tilde{f}_t\left|\mathcal{F}_{t-1}\right.\right]^T \tilde{\phi}_{t-1}},$$

$$\beta^T_{k,t-1} \times -R^f_{t-1} cov\left(f_t, M(t)\left|\mathcal{F}_{t-1}\right.\right),$$

which by using the definition of covariance in terms of expectations gives,

$$E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right] - r^f_{t-1} = \beta^T_{k,t-1} \lambda_{t-1}, \tag{4}$$

where $\beta^T_{k,t-1} = \left[cov\left(f_t, f^T_t\left|\mathcal{F}_{t-1}\right.\right)\right]^{-1} cov\left(f_t, r_{k,t}\left|\mathcal{F}_{t-1}\right.\right)$; and λ_{t-1} is a vector with element

$$\lambda_{n,t-1} = -R^f_{t-1}\left(E\left[\left[f_{n,t} M(t)\right]\left|\mathcal{F}_{t-1}\right.\right] - E\left[M(t)\left|\mathcal{F}_{t-1}\right.\right] E\left[f_{n,t}\left|\mathcal{F}_{t-1}\right.\right]\right) \forall n = 1, \ldots, N.$$

Note that the conditional moment restrictions (2) and (4) are together equivalent to the moment condition,

$$E\left[M(t) R_{k,t}\left|\mathcal{F}_{t-1}\right.\right] = 1, \forall k \ and \tag{5}$$

$$\lambda_{n,t-1} = E\left[f_{n,t}\left|\mathcal{F}_{t-1}\right.\right] - R^f_{t-1} E\left[M(t) f_{n,t}\left|\mathcal{F}_{t-1}\right.\right] \quad \forall n = 1, \ldots, N. \tag{6}$$

If f_n is an excess return then,

$$E\left[M\left(t\right)f_{n,t}\left|\mathcal{F}_{t-1}\right.\right]=0,\ \forall n=1,\ldots,N,$$

and $\lambda_{n,t-1}$ is the conditional mean of the nth risk factor. If f_n is a simple return then $E\left[M\left(t\right)f_{n,t}\left|\mathcal{F}_{t-1}\right.\right]=1,\ \forall n=1,\ldots,N$ and (6) gives excess returns.

We say that the nth risk factor is "priced" if $\lambda_{n,t-1}>0$. If $\lambda_{n,t-1}=0$ then it must be that $M(t)$ and $f_{n,t}$ are uncorrelated. Cochrane (1996) notes that finding that the nth risk factor is priced is not the same as finding that the nth risk factor is useful in pricing assets i.e., $\phi^f_{n,t}\left(t\right)>0\ \forall t$ in equation (1).

Example. (The conditional CAPM) Let $N=1$ and

$M\left(t\right)=\phi^0_{t-1}+\phi^m_{t-1}\left(r^m_t-r^f_{t-1}\right)$ where r^m_t is the return of some benchmark market index. Then substituting the pricing kernel into (6) and the resulting expression into (4) gives,

$$E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right]-r^f_{t-1}=\beta_{k,t-1}\left(E\left[r_{m,t}\left|\mathcal{F}_{t-1}\right.\right]-r^f_{t-1}\right),\ \forall r_{k,t}\in\mathbb{R}_{k,t}.$$

Example. (The multifactor conditional asset pricing model) Let $M\left(t\right)=g\left(r^{P1},\ldots,r^{PN}\right)$ where g is some affine function of N benchmark returns. Then for any $r_{k,t}\in\mathbb{R}_{k,t}$,

$$\begin{aligned}E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right]-r^f_{t-1}&=\beta^1_{k,t-1}\left(E\left[r^1_t\left|\mathcal{F}_{t-1}\right.\right]-r^f_{t-1}\right)+\cdots\\&+\beta^N_{k,t-1}\left(E\left[r^N_t\left|\mathcal{F}_{t-1}\right.\right]-r^f_{t-1}\right),\ \forall r_{k,t}\in\mathbb{R}_{k,t}.\end{aligned}$$

Note. The conditional factor model represented in reduced-form should have a "connection" with some underlying structural economic model in order to price all payoffs (returns) in the economy.

2. Conditioning Down the Conditional CAPM

Let $N=1$ and $f_t=r^m_t-r^f_{t-1}$.

A. If $\mathcal{F}_t$ is the null information set then,

$$E\left[r_{k,t}\right]-\mu_0=\beta_kE\left[f_t\right],\ \forall k$$

where $E\left[\tilde{\phi}_t\right]\mu_0=1$; $\beta_k=\frac{cov(r_{k,t},f_t)}{Var(f_t)}$; and $\phi_{t-1}=\phi$ are state independent, which is not consistent with dynamic economies.

B. More generally, we fix an information set $\mathcal{F}_t$ and assume $r^f_{t-1}\in\mathcal{F}_{t-1}$ and $r^m_t\in\mathcal{F}_t$. Recall $E\left[r_{k,t}\left|\mathcal{F}_{t-1}\right.\right]-r^f_{t-1}=\beta^{\mathcal{F}}_{k,t-1}\lambda^{\mathcal{F}}_{t-1}$, where $\lambda^{\mathcal{F}}_{t-1}=E\left[r^m_t-r^f_{t-1}\left|\mathcal{F}_{t-1}\right.\right]$. Define $R_t\equiv\left(r_t-r^f_{t-1}\right)$ for any r_t, then the equilibrium restriction is the pair $\left(\alpha_{k,t-1},\beta^{\mathcal{F}}_{k,t-1}\right)$ that solves the conditional least-squares minimization problem,

$$\underset{\left\{\alpha_{k,t-1},\beta^{\mathcal{F}}_{k,t-1}\right\}}{Min}E\left[\left(R_t-\alpha_{k,t-1}-\beta^{\mathcal{F}}_{k,t-1}R^m_t\right)\left|\mathcal{F}_{t-1}\right.\right],\tag{7}$$

satisfies,

$$\beta^{\mathcal{F}}_{k,t-1}=\frac{cov\left[\left(r_{k,t},r^m_t\right)\left|\mathcal{F}_{t-1}\right.\right]}{Var\left(r^m_t\left|\mathcal{F}_{t-1}\right.\right)},\ and\ \alpha_{k,t-1}=E\left[R_{k,t}\left|\mathcal{F}_{t-1}\right.\right]-\beta^{\mathcal{F}}_{k,t-1}E\left[R^m_t\left|\mathcal{F}_{t-1}\right.\right]=0.$$

The conditional beta model implies that the conditional alphas are zero, but this don't imply that the unconditional alphas are zero too! Define $\beta_k\equiv E\left[\beta^{\mathcal{F}}_{k,t-1}\right]$ as the mean of the conditional beta for each security k, and let $\xi_{k,t-1}$ denote the zero-mean random deviation of the conditional beta from its mean $\xi_{k,t-1} \equiv \beta^{\mathcal{F}}_{k,t-1} - \beta_k$. If we substitute $\beta^{\mathcal{F}}_{k,t-1}$ into the F.O.N.C. of problem (7) and condition down to unconditional expectations we verify that,

$$E\left[\left(R_{k,t} - \beta_k R^m_t\right) R^m_t - \xi_{k,t-1}\left(R^m_t\right)^2\right] = 0, \; and \tag{8}$$

$$E\left[\left(R_{k,t} - \beta_k R^m_t\right) - \xi_{k,t-1} R^m_t\right] = 0. \tag{9}$$

If,

$$E\left[\xi_{k,t-1}\left(R^m_t\right)^2\right] = 0, \; and \tag{10}$$

$$E\left[\xi_{k,t-1} R^m_t\right] = 0, \tag{11}$$

then (8) and (9) reduce to the normal equations of the unconditional least-squares projection. That is, $\alpha_k = E\left[R_{k,t}\right] - \beta_k E\left[R^m_t\right] = 0$ and we obtain the unconditional CAPM with market beta $\beta_k \equiv E\left[\beta^{\mathcal{F}}_{k,t-1}\right]$. Conditions (10) and (11) to condition down the conditional CAPM can be re-written as,

$$E\left[\xi_{k,t-1}\left(R^m_t\right)^2\right] = cov\left(\xi_{k,t}, \sigma^2_{m,t-1}\right) + cov\left(\xi_{k,t-1}, \left(\lambda^{\mathcal{F}}_{t-1}\right)^2\right) = 0, \; and \tag{12}$$

$$E\left[\xi_{k,t-1} R^m_t\right] = cov\left(\xi_{k,t-1}, \lambda^{\mathcal{F}}_{t-1}\right) = 0, \tag{13}$$

which are Lewellen and Nagel (2006) conditions for the unconditional CAPM. That is, any variation in $\beta^{\mathcal{F}}_{k,t-1}$ should be uncorrelated with both the market price of risk $\left(\lambda^{\mathcal{F}}_{t-1}\right)$ and with $\left(\sigma^2_{m,t-1} + \lambda^{\mathcal{F}}_{t-1}\right)$.

The empirical literature using the conditional CAPM generally assumes that $M(t)$ is some affine function of a state vector z_{t-1} with coefficients,

$$\phi^0_{t-1} = a^0 + b^0 z_{t-1}, \; and \; \phi^f_{t-1} = a^f + b^f z_{t-1},$$

for some $z_t \in \mathcal{F}_t$, which is observable by the investors but not by the econometrician. Assume $N = 1$ then,

$$E\left[\left(a^0 + b^0 z_{t-1} + a^f f_t + b^f f_t z_{t-1}\right) R_{k,t}\right] = 1, \; \forall k. \tag{14}$$

Let $f^*_t \equiv \left(f_t, z_{t-1}, f_t z_{t-1}\right)$ denote the expanded set of risk factors, and $M^*(t) \equiv a^0 + b^0 z_{t-1} + a^f f_t + b^f f_t z_{t-1}$ the pricing kernel or stochastic discount factor. Then (14) is treated as an expanded unconditional version of the conditional CAPM with,

$$E\left[r_{k,t}\right] - \mu = \beta^T_k \lambda, \; \forall k$$

where μ is the return on an unconditional zero-beta portfolio; $\beta_k = \frac{cov\left(f^*_t, r_{k,t}\right)}{cov\left(f^*_t, f^{*T}_t\right)}$ is a 3×1 vector; and $\lambda = -\mu cov\left(f^*_t, M^*(t)\right)$ is a 3×1 vector too.

If (12) and (13) don't hold, then for the econometrician that is unable to observe the conditional information used by investors to price assets (i.e,. her filtration set is the null set) any variables useful to predict the state-dependent λ_t should appear as additional factors in the unconditional version of the model.

References

[Cochrane (1996) Political Economy]
[Lewellen and Nagel (2006) Financial Economics]