# LECTURE NOTES 5

ARIEL M. VIALE

## 1. The Martingale Approach to Asset Pricing in Continuous Time

1.1. **Money market state price deflator.** Let $r(t)$ be the instantaneous risk-free rate at date $t$ such that $R(t) \equiv R(0)\, e^{\int_0^t r(s)ds}$. Thus, by Ito's lemma $\frac{dR}{R} = r(t)\,dt$ given $R(0) = 1$. Assuming there exists a SDF to price contingent claims at date $T$, then the price of any random payoff $\tilde{X}$ received at date $T$ can be appropriately deflated to make it driftless i.e., a martingale. Define the money market state price deflator,

$$1 \times M(t) \equiv E\left[e^{\int_t^T r(s)ds}\tilde{\phi}\right], \quad where\ \tilde{\phi} = M(T).$$

*Claim.* i) Price at time $t = 0$ of any random payoff $\tilde{X}$ at date $t \leq T$ is $P_0\left(\tilde{X}\right) = E_0\left[M(t)\,\tilde{X}\right]$; ii) Price at date $t \leq T$ of any random payoff $\tilde{X}$ received at time $T$ is $P_t\left(\tilde{X}\right) = E_t\left[\frac{M(T)}{M(t)}\tilde{X}\right] = \frac{1}{M(t)}E_t\left[M(T)\,\tilde{X}\right]$.

*Proof.* i) Price at time $t = 0$ of receiving $\tilde{X}$ at time $t \leq T$ is equal to the price at time $t = 0$ of receiving $e^{\int_t^T r(s)ds}\tilde{X}$ at $T$, which is equal to $E_0\left[M(T)\,e^{\int_t^T r(s)ds}\tilde{X}\right]$ $= E_0\left[E_t\left\{M(T)\,e^{\int_t^T r(s)ds}\tilde{X}\right\}\right]$ by the law of iterated expectations. Thus,
$E_0\left[\tilde{X}E_t\left\{M(T)\,e^{\int_t^T r(s)ds}\right\}\right] = E_0\left[\tilde{X}M(t)\right]$ as required.

ii) Let $\tilde{\phi}_{t,T} = SDF$ at date $t$ for contingent claims with maturity $T$, meaning price at $t$ of receiving $\tilde{X}$ at $T$ is equal to $E_t\left[\tilde{\phi}_{t,T}\tilde{X}\right] = E\left[M(t)\,E_t\left[\tilde{\phi}_{t,T}\tilde{X}\right]\right] = E\left[E_t\left[M(t)\,\tilde{\phi}_{t,T}\tilde{X}\right]\right] = E\left[M(t)\,\tilde{\phi}_{t,T}\tilde{X}\right]$, by law of iterated expectations. Recall that $M(t)\,\tilde{\phi}_{t,T} = SDF$ at time $t = 0$, hence its equal to $M(T) \Rightarrow \tilde{\phi}_{t,T} = \frac{M(T)}{M(t)}$ as required. □

1.2. **Equivalent martingale measure.** From 1.1. is clear that we only need to know $M(t,T)$ to price assets. Let $\frac{dS}{S} = \mu dt + \sigma dB$, $\frac{dV}{V} = (\mu + q)\,dt + \sigma dB$, and $M(t) = e^{r(T-t)}E\left[\tilde{\phi}\right]$ with $\tilde{\phi} = M(T)$. Then,

*Claim.* 1. $M(t) = e^{r(T-t)}Y(t)$, where $Y(t)$ is a martingale.

*Proof.* $\forall t < u$, $Y(u) = E_u\left[\tilde{\phi}\right] \Rightarrow E_t\left[Y(u)\right] = E_t\left[E_u\left[\tilde{\phi}\right]\right] = E_t\left[\tilde{\phi}\right] = Y(t)$ as required. □

*Claim.* 2. The deflated process $MV$ is a martingale.

*Date*: 09/29/09 - Rev.: 10/1/13.

*Proof.* $E_t[M(u)V(u)] = E_t[M(u)e^{qu}S(u)] = e^{qu}E_t[M(u)S(u)] =$
$\frac{1}{M(t)}E_t[M(u)S(u)] = e^{-q(u-t)}S(t) =$ price at time $t$ of getting $S(u)$ at time $u > t \Rightarrow e^{qu}E_t[M(u)S(u)] = e^{qt}M(t)S(t) = M(t)V(t)$ as required. □

*Claim.* 3. $\frac{dY}{Y} = -\theta dB$ for some stochastic process $\theta$.

*Proof.* (**Martingale representation theorem**) Assume $dlogY = -\frac{1}{2}\theta^2 dt - \theta dB \Rightarrow logY(t) = logY(0) - \int_0^T \frac{1}{2}\theta^2(t)\,dt - \int_0^T \theta(t)\,dB(t)$

$\Rightarrow Y(T) = Y(0)\,e^{-\int_0^T \frac{1}{2}\theta^2(t)dt - \int_0^T \theta(t)dB(t)}$. Recall that $M(T) = Y(T)$ and $Y(0) = E\left[\tilde{\phi}\right]$.

Thus, $M(t) = e^{r(T-t)}Y(t) = e^{rT}e^{-rt}Y(t) \Rightarrow \frac{dM}{M} = -rdt + \frac{dY}{Y}$ by Ito's lemma. That is, $\frac{dM}{M} = -rdt - \theta dB$.

The deflated process satisfies,

$$\frac{d(MV)}{MV} = \frac{dM}{M} + \frac{dV}{V} + \left(\frac{dM}{M}\right)\left(\frac{dV}{V}\right), \quad by\ Ito's\ lemma$$
$$= (-r + \mu + q - \theta\sigma)\,dt - \theta dB + \sigma dB$$
$$\Rightarrow -r + \mu + q - \theta\sigma = 0,$$

because $MV$ is a martingale by claim #2 i.e., it is a driftless process and, □

**Definition 1.** (**Market price of risk**) $\theta \equiv \frac{\mu+q-r}{\sigma}$ is the unique market price of risk (Sharpe's ratio on stocks).

### 1.3. Feynman-Kac solution.

**Proposition.** *The unique SDF is the Feynman-Kac solution to the PDE of the deflated stochastic process,*

$$M(T) = e^{-rT - \frac{1}{2}\theta^2 T - \theta B(T)}. \tag{1.1}$$

**Example.** Price of a call at date $t = 0 \Rightarrow C(0) = E\left[M(T)(0, S(T) - K)^+\right]$, where $M(T) = e^{-\int_t^T r(s)ds}$ is the Feynman-Kac solution to the BSM PDE (3).

1.4. **Fundamental Asset Pricing Equation.** Let $M(t) = E_t\left[e^{\int_t^T r(s)ds}\tilde{\phi}\right]$ be the state price deflator, $V$ is the price of a non-dividend paying portfolio, so that the deflated process $MV$ is a martingale. Set $M(t) = e^{-\int_0^t r(s)ds}E_t\left[e^{\int_0^T r(s)ds}\tilde{\phi}\right]$, and define $Y(t) \equiv E_t\left[e^{\int_0^T r(s)ds}\tilde{\phi}\right]$ so that $M(t) = e^{-\int_0^t r(s)ds}Y(t)$. Applying Ito's lemma twice we obtain,

$$\frac{dM}{M} = -rdt + \frac{dY}{Y},\ and$$
$$\frac{d(MV)}{MV} = \frac{dM}{M} + \frac{dV}{V} + \left(\frac{dM}{M}\right)\left(\frac{dV}{V}\right),$$

which has no drift. Notice that $\frac{dV}{V}$ does have drift (expected return) and,

$$0 = \frac{d(MV)}{MV} = (\mu + q - r)\,dt + \left(\frac{dM}{M}\right)\left(\frac{dV}{V}\right),$$

$$(\mu + q - r)\,dt = -\left(\frac{dM}{M}\right)\left(\frac{dV}{V}\right). \tag{1.2}$$

**Example.** For a static one-period discrete asset pricing model we show that
$risk\ premium = -R_f cov\left(\tilde{R}_i, \tilde{\phi}\right)$.

**Example.** For the BSM model, $\theta = \frac{\mu+q-r}{\sigma}$, $\frac{dM}{M} = -rdt - \theta dB$, $\frac{dV}{V} = (\mu+q)\,dt + \sigma dB \Rightarrow \left(\frac{dM}{M}\right)\left(\frac{dV}{V}\right) = -\theta\sigma dt$ and $\therefore (\mu+q-r)\,dt = -\left(\frac{dM}{M}\right)\left(\frac{dV}{V}\right)$.

1.5. **General asset pricing with multiple assets and sources of risk assuming markets are (dynamically) complete.**

$$\underset{c}{Max} E\left[u(c)\right],$$

$$s.t.\ E\left[\tilde{\phi}c\right] = W(0),$$

where $W(0)$ is some given initial level of wealth; and $u(c)$ is some utility function satisfying some technicalities (i.e., quasiconcave, SI, TI, and bounded). Note that $u'(c) = \theta\tilde{\phi}$ for some constant $\theta \Rightarrow \frac{1}{\theta}u'(c) = SDF$. Let $R_1$, $R_2$ be returns on portfolio strategies e.g.,

$$\begin{array}{c} R_1 = \frac{e^{qT}S(T)}{S(0)} \\ R_2 = e^{rT} \end{array},$$

where the dividend yield and risk free rate are not necessarily constant.

*Claim.* Let $c$ be optimal consumption. Then markets are complete and it must be,

$$E\left[u'(c)\,R_1\right] = E\left[u'(c)\,R_2\right].$$

*Proof.* (b.w.o.c.) Suppose not. W.l.o.g. assume that the direction of improvement in utility is as follows,

$$E\left[u'(c)\,(R_1 - R_2)\right] > 0.$$

Define $c^* = c + \varepsilon \times \underbrace{(R_1 - R_2)}_{zero\ cost\ and\ feasible}$ for some $c > 0$. Then $u(c^*) = u(c) + u'(\hat{c})\,\varepsilon\,(R_1 - R_2)$ for some $\hat{c}$ between $c$ and $c^*$ by Mean-value theorem. Thus,

$$E\left[u(c^*)\right] = E\left[u(c)\right] + \varepsilon E\left[u'(c^*)\,(R_1 - R_2)\right] > E\left[u(c)\right],$$

for some sufficiently small $\varepsilon$. Set $k = E\left[u'(c)\,R_1\right]$ and $E\left[\frac{1}{k}u'(c)\,R\right] = 1\ \forall R \Longrightarrow \frac{1}{k}u'(c) = SDF$. Note that there will be a different SDF for each investor according to their marginal utilities $u'(c)$ and markets are incomplete. □

Define $M(T) = \frac{1}{k}u'(c)$. Set $Y(T) = M(T), Y(t) = E_t\left[Y(T)\right], M(t) = E_t\left[e^{\int_t^T r(s)ds}M(T)\right]$, *and* $W(T) = c$. Then $W(t)\,M(t) = E_t\left[W(T)\,M(T)\right]$.

**Fact.** *For any non-dividend paying portfolio with price $V$, the state deflated process $MV$ is a martingale by martingale representation theorem.*

**Fact.** *(Risk premium on $V$)$dt = -\left(\frac{dV}{V}\right)\left(\frac{dM}{M}\right)$, and we need wealth to be correlated with the state price deflator.*

Suppose $n$ Brownian motions. Then, $\frac{dM}{M} = -rdt - \sum_{i=1}^{n}\theta_i dB_i$, where $\theta_i$ can be stochastic. Define $\theta = \sqrt{\sum_{i=1}^{n}\theta_i^2}$ and note that $dB = \frac{\sum_i \theta_i dB_i}{\sqrt{\sum_i \theta_i^2}}$ is a Brownian motion by Levy's theorem (i.e., $(dB)^2 = \frac{\sum\theta^2 dt}{\sum\theta^2} = dt$). Thus for any asset, $\frac{dV}{V} = something\ dt + \sum_i \sigma_i dB_i$ and $risk\ premium \cdot dt = \sum_{i=1}^{n}\theta_i\sigma_i$. Also,

$$\frac{\sum_{i=1}^{n}\theta_i\sigma_i}{\sqrt{\sum_i \sigma_i^2}},$$

is Sharpe's ratio. Note that the squared Sharpe's ratio is equal to,

$$\frac{\left(\sum_{i=1}^{n}\theta_i\sigma_i\right)^2}{\sum_i\sigma_i^2}\leq\frac{\left(\sum_{i=1}^{n}\sigma_i^2\right)\left(\sum_{i=1}^{n}\theta_i^2\right)}{\sum_i\sigma_i^2}=\sum_{i=1}^{n}\theta_i^2,$$

where $\theta^2$ is Hansen-Jagannathan's lower bound. Recall $\frac{dY}{Y}=-\theta dB$ and assume that $\theta(t)$ depends only on $B(s)$ for some $s\leq t$. As $M(T)W(T)$ depends on $Y(T)$ then $W(T)=c$ must satisfy $\frac{1}{k}u'(W(T))=Y(T)$. By the martingale representation theorem,

$$\frac{d(MW)}{MW}=\rho dB,$$
$$\Rightarrow\frac{dM}{M}+\frac{dW}{W}+\left(\frac{dM}{M}\right)\left(\frac{dW}{W}\right)=\rho dB,$$
$$\frac{dW}{W}=something\ dt+(\rho+\theta)\,dB.$$

Note that if $\frac{dW}{W}$ perfectly correlates with $dB$ then we can substitute $W$ with $M$ such that,

$$something\ dt=-risk\ premium\ of\ W. \tag{1.3}$$

and $\frac{dM}{M}=-\theta\frac{dW}{W}$ (by Martingale representation theorem).

**Example.** Multiple Assets Pricing Formula: The instantaneous CAPM assuming myopic investors and log-normal returns.

Let,

$$\frac{dS_k}{S_k}=\mu_k dt+\sum_{i=1}^{n}\sigma_{k,i}dB_i,\forall k=1,\ldots,m,\ and\ i=1,\ldots,n\ (independent).$$

Given a SDF assume markets are complete ($m\geq n$). Define $M(t)\equiv e^{r(T-t)}E_t\left[\tilde{\phi}\right]$, $Y(t)=E_t\left[\tilde{\phi}\right]$ is a martingale. That is, by the martingale representation theorem $\Rightarrow\frac{dY}{Y}=-\sum_{i=1}^{n}\theta_i dB_i\Rightarrow\frac{dM}{M}=-rdt-\sum_{i=1}^{n}\theta_i dB_i$. Assume $\theta_i's$ are constant. Then, for a RA with log utility preferences the portfolio choice problem is,

$$\underset{c}{Max}E\left[log\ c\right],$$
$$s.t.\ E\left[\tilde{\phi}c\right]=W(0),$$

for some given initial value for wealth. Assuming interior solutions exists the F.O.N.C.s w.r.t. $c$ are,

$$c=\frac{W(0)}{\tilde{\phi}},$$
$$M(t)W(t)=E_t\left[M(T)W(T)\right]=W(0),$$
$$\Rightarrow W(t)=W(0)e^{rt+\frac{1}{2}\sum_{i=1}^{n}\theta_i^2 t+\sum_{i=1}^{n}\theta_i B_i(t)},by\ Ito's\ lemma,$$
$$\Rightarrow\frac{dW}{W}=rdt+\sum_{i=1}^{n}\theta_i^2 dt+\sum_{i=1}^{n}\theta_i dB_i,by\ Ito's\ lemma.$$

Find portfolio $\pi$. Thus,

$$\frac{dW}{W}=rdt+\sum_{k=1}^{m}\pi_k\left(\mu_k+q_k-r\right)dt+\sum_{k=1}^{m}\pi_k\sum_{i=1}^{n}\sigma_{k,i}dB_i,$$

$$= rdt + \sum_{k=1}^{m} \pi_k \left(\mu_k + q_k - r\right) dt + \sum_{k=1}^{m} \sum_{i=1}^{n} \pi_k \sigma_{k,i} dB_i.$$

So $\sum_{i=1}^{n} \pi_k \sigma_{k,i} = \theta_i, \quad for\, k = 1, \ldots, m \Leftrightarrow (\sigma_{1,i}, \sigma_{2,i}, \ldots, \sigma_{m,i}) \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_m \end{pmatrix} = \theta_i.$

Set $\sum^T = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nm} \end{pmatrix}$, and solve $\underbrace{\sum}_{n\times m}^{T} \underbrace{\pi}_{m\times 1} = \underbrace{\begin{pmatrix} \theta_1 \\ \vdots \\ \theta_n \end{pmatrix}}_{n\times 1}$. Thus,

$\begin{pmatrix} \frac{dS_1}{S_1} \\ \vdots \\ \frac{dS_m}{S_m} \end{pmatrix} = \frac{dS}{S} = \mu dt + \sum \begin{pmatrix} dB_1 \\ \vdots \\ dB_n \end{pmatrix}$ with variance-covariance matrix $\left(\frac{dS}{S}\right)\left(\frac{dS}{S}\right)^T \Leftrightarrow \sum (dB)(dB)^T \sum^T = \sum \sum^T dt$, and $\pi = \left(\sum \sum^T\right)^{-1} \sum \theta$. Put $V_k(t) = e^{q_k t} S_k(t)$. Then, for each $k$ $MV_k$ is a martingale and,

$$\frac{dM}{M} + \frac{dV_k}{V_k} + \left(\frac{dM}{M}\right)\left(\frac{dV_k}{V_k}\right) \; has\; no\; drift.$$

$$\Rightarrow (\mu_k + q_k)\, dt = rdt - \left(\frac{dM}{M}\right)\left(\frac{dV_k}{V_k}\right),$$

$$and\; \left(\frac{dM}{M}\right)\left(\frac{dV_k}{V_k}\right) = -\theta \left(\frac{dW}{W}\right)\left(\frac{dV_k}{V_k}\right),$$

$$\underbrace{\Rightarrow (\mu_k + q_k - r)\, dt = risk\; premium\; W \;\times \underbrace{\frac{\left(\frac{dW}{W}\right)\left(\frac{dV_k}{V_k}\right)}{\left(\frac{dW}{W}\right)^2}}_{\frac{cov(\cdot,W)}{var(W)}}}_{instantaneous\; CAPM}. \tag{1.4}$$

## References

[Back (1991) Mathematical Economics]
[Chamberlain (1987) Econometrica]