LECTURE NOTES 4

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1. Brownian Motion

Definition 1. (**Brownian motion**) A random process that evolves continuously in time and has the property that its change over any period of time is normally distributed with mean zero and variance equal to the length of the time period.

Note that the brownian motion is a martingale with continuous paths. Let $B(t)$ denote the value of the Brownian motion at time t, then for any $u > t$ given the information at time t it must be that $B(u) \sim N(B(t), u-t) \Leftrightarrow B(u) - B(t) \sim N(0, u-t)$.

Definition 2. (**Quadratic variation**) Consider the discrete partition,

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = T,$$

where N is the number of time intervals $\Delta t = t_i - t_{i-1} \forall i$ in the path. Let $\sum_{i=1}^{N} [B(t_i) - B(t_{i-1})]^2 = \sum_{i=1}^{N} [\Delta B(t_i)]^2$. Consider a finer partition of the path as $N \to \infty$, then $\Delta t = t_i - t_{i-1} \to 0$ and $\sum_{i=1}^{N} [\Delta B(t_i)]^2 \to T \quad a.s.$

Claim. Notice that the Brownian motion is a very unusual continuous function. The usual continuously differentiable function has quadratic variation equal to zero.

Proof. Let $X(t) = at$ for some constant a. Compute the quadratic variation $\sum_{i=1}^{N} [\Delta X(t_i)]^2 = \sum_{i=1}^{N} [at_i - at_{t-i}]^2 = \sum_{i=1}^{N} [a(t_i - t_{t-i})]^2$
$= \sum_{i=1}^{N} [a\Delta t]^2 = a^2 \sum_{i=1}^{N} \left[\frac{T}{N}\right]^2 = a^2 N \left(\frac{T}{N}\right)^2 = \frac{a^2T^2}{N} \to 0$ as $N \to \infty$. □

Definition 3. (**Total variation**) $\sum_{i=1}^{N} |B(t_i) - B(t_{i-1})| = \sum_{i=1}^{N} |\Delta B(t_i)| \to \infty \quad a.s.$

Why Brownian motion?

Because as we already learned, asset pricing involves martingales (Recall the Binomial model?). Furthermore, continuous processes are much more mathematically tractable than e.g., jump processes.

Theorem. *(Levy's theorem) A continuous martingale is a Brownian motion if and only if its quadratic variation over each interval* $[0, T]$ *equals* T.

2. Ito (Difussion) Processes

Definition 4. (**Ito process**) Is a random variable X that changes over time as,

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dB(t), \tag{2.1}$$

where B is a Brownian motion, and μ and σ can also be random processes known at time t.

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Adding changes over time gives,

$$X(T) = X(0) + \int_0^T \mu(t)\,dt + \int_0^T \sigma(t)\,dB(t)\,, \tag{2.2}$$

for any $T > 0$. An Ito process evolves continuously over time with drift $\mu(t)$ and difussion coefficient $\sigma(t)$ s.t. $\int_0^T \sigma(s)\,dB(s) \approx \sum_{i=1}^N \sigma(t-i)\left[B(t_i) - B(t_{i-1})\right] \forall i$.

Intuition: Let X be some asset price (no dividends), and θ the # shares invested in the asset portfolio. Then, the value of the asset portfolio is $\int_0^T \theta(t)\,dX(t) \approx \sum_{i=1}^N \theta(t-i)\left[X(t_i) - X(t_{i-1})\right] \quad \forall i$.

If we assume $\mu = 0$ and $E\left[\int_0^T \sigma^2(t)\,dt\right] < \infty$ for each T, then the Ito process is a continuous martingale with finite variance.

Claim. Independently of $\mu = 0$ and $var\left[X(t)\right] < \infty$ the quadratic variation of a difussion process X is $\underset{N\to\infty}{lim} \sum_{i=1}^N \left[\Delta X(t_i)\right]^2 = \int_0^T \sigma^2(t)\,dt \quad a.s.$

Proof. Notice that $(dt)^2 = 0$, $(dt)(dB) = 0$, $(dB)^2 = dt$. If $dX = \mu dt + \sigma dB$ for some Brownian motion B, then $(dX)^2 = (\mu dt + \sigma dB)^2$
$\Rightarrow \mu^2 \underbrace{(dt)^2}_{=0} + 2\mu\sigma \underbrace{(dt)(dB)}_{=0} + \sigma^2 (dB)^2 = \sigma^2 dt$. Integrating this from 0 to T gives the quadratic variation,

$$\int_0^T (dX(t))^2 = \int_0^T \sigma^2(t)\,dt. \tag{2.3}$$

□

In conclusion difussion processes are a very special type of continuous martingales with different quadratic variation than a Brownian motion.

3. Ito's Lemma

Review: Given two continuously differentiable functions $y = f(x)$ and $x = g(t)$ such that $y = f(g(t))$, what is $\frac{dy}{dt}$? By the chain formula of standard calculus,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} = f'(x(t)) \times g'(t)\,.$$

Over $[0, T]$ the latter gives,

$$y(T) - y(0) = \int_0^T \frac{dy}{dt} dt = \int_0^T f'(x(t)) \times g'(t)\,dt = \int_0^T f'(x(t)) \times dx(t)$$

$$\approx \sum_{i=1}^N f'(x(t_i))\left[x(t_i) - x(t_{i-1})\right].$$

Definition 5. (**Ito's lemma**) We define $Y = f(B(t))$ where B is a Brownian motion. Thus, $dY = f'(B(t))\,dB + \frac{1}{2} f''(B(t))\,dt$ and $Y(T) - Y(0) = \int_0^T f'(B(t))\,dB(t) + \frac{1}{2}\int_0^T f''(B(t))\,dt$
$\approx \sum_{i=1}^N f'(B(t_i))\left[B(t_i) - B(t_{i-1})\right] + \frac{1}{2}\int_0^T f''(B(t))\,dt$.

Intuition: Recall from ordinary calculus that the derivative defines a linear approximation of the change in Y over some time period. A better approximation is given by the second order Taylor series expansion,

$$\Delta y \approx f'(x)\,\Delta x + \frac{1}{2} f''(x)\,(\Delta x)^2,$$

$$y(T) - y(0) \approx \sum_{i=1}^{N} f'(x(t_i))\,\Delta x + \underbrace{\frac{1}{2}\sum_{i=1}^{N} f''(x(t_i))\,(\Delta x)^2}_{\to 0 \; as \; N \to \infty}.$$

In Ito's calculus this is,

$$\Delta Y = f'(B)\,\Delta B + \frac{1}{2} f''(B)\,(\Delta B)^2,$$

Recall $(dB)^2 = dt$ so $(\Delta B)^2 \approx dt$ as
$\underset{N\to\infty}{lim} \sum_{i=1}^{N} [\Delta B(t_i)]^2 = \int_0^T dt \Rightarrow \frac{1}{2}\int_0^T f''(B(t))\,dt \approx \frac{1}{2}\sum_{i=1}^{N} f''(B(t_i))\,(\Delta B)^2.$

Example. Let $Y = log(X)$ by Ito's lemma,

$$dY = f'(X)\,dX + \frac{1}{2} f''(X)\,(dX)^2,$$

$$f'(X) = \frac{1}{X}, \quad f''(X) = -\frac{1}{X^2},$$

$$\Rightarrow dY = \frac{dX}{X} - \frac{1}{2}\left(\frac{dX}{X}\right)^2.$$

Example. Let $Y = e^X$ by Ito's lemma,

$$dY = f'(X)\,dX + \frac{1}{2} f''(X)\,(dX)^2,$$

$$f'(X) = e^X, \quad f''(X) = e^X,$$

$$\frac{dY}{Y} = dX + \frac{(dX)^2}{2}.$$

Example. Let $Z = XY$. Define $g(x,y) = xy$. Notice that $dZ = \frac{\partial g}{\partial x} dX + \frac{\partial g}{\partial y} dY + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(dY)^2 + \frac{\partial^2 g}{\partial x \partial y}(dX)(dY)$ by Ito's lemma for multiple processes. And $\frac{\partial g}{\partial x} = y$, $\frac{\partial g}{\partial y} = x$, $\frac{\partial^2 g}{\partial x^2} = 0 = \frac{\partial^2 g}{\partial y^2}$, $\frac{\partial^2 g}{\partial x \partial y} = 1$. Thus,

$$\Rightarrow dZ = Y dX + X dY + (dX)(dY) \Rightarrow \frac{dZ}{Z} = \frac{dX}{X} + \frac{dY}{Y} + \left(\frac{dX}{X}\right)\left(\frac{DY}{Y}\right).$$

Example. Let $Z = \frac{Y}{X}$. Define $g(x,y) = x^{-1}y$. Notice that $dZ = \frac{\partial g}{\partial x} dX + \frac{\partial g}{\partial y} dY + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(dX)^2 + \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(dY)^2 + \frac{\partial^2 g}{\partial x \partial y}(dX)(dY)$ by Ito's lemma for multiple processes. And $\frac{\partial g}{\partial x} = -\frac{y}{x^2}$, $\frac{\partial g}{\partial y} = \frac{1}{x}$, $\frac{\partial^2 g}{\partial x^2} = \frac{y}{x^3}$, $\frac{\partial^2 g}{\partial y^2} = 0$, $\frac{\partial^2 g}{\partial x \partial y} = -\frac{1}{x^2}$. Thus,

$$\Rightarrow dZ = -\frac{Y}{X^2} dX + \frac{1}{X} dY + \frac{1}{2}\frac{Y}{X^3}(dX)^2 - \frac{1}{X^2}(dX)(dY),$$

$$\frac{dZ}{Z} = \frac{dY}{Y} - \frac{dX}{X} + \frac{1}{2}\left(\frac{dX}{X}\right)^2 - \left(\frac{dY}{Y}\right)\left(\frac{dX}{X}\right).$$

4. Geometric Brownian Motion

We assume $\frac{dS}{S}=\mu dt+\sigma dB \Leftrightarrow dS=\mu Sdt+\sigma SdB$. Set $Y\left(t\right)=logS\left(t\right)=g\left(S\left(t\right)\right)$ where $g=log$. Thus,

$$dY=\frac{dS}{S}+\frac{1}{2}\left(-\frac{1}{S^2}\right)\left(dS\right)^2$$

$$=\mu dt+\sigma dB-\frac{1}{2S^2}\sigma^2 S^2 dt$$

$$=\left(\mu-\frac{1}{2}\sigma^2\right)dt+\sigma dB.$$

Suppose S = asset price without dividends. Let $u>t$ so that $logS\left(u\right)-logS\left(t\right)$ =continuously compounded return= $r\left(u-t\right)$. Then, $S\left(u\right)=S\left(t\right)e^{r(u-t)}$ and the continuously compounded rate of return over a period of length Δt is given by,

$$r_i=\underbrace{\mu-\frac{1}{2}\sigma^2}_{mean}+\frac{\sigma}{\Delta t}\Delta B,$$

with variance $\frac{\sigma^2}{\Delta t}$.

Suppose now that the asset pays a constant instantaneous (at instant dt) dividend yield $qS\left(t\right)dt$. Let $X\left(t\right)$ denote the number of shares invested in the asset at time t, and assume reinvesting of dividends in more shares. That is, $dX\left(t\right)=\frac{X(t)qS(t)dt}{S(t)}=qX\left(t\right)dt$ and $\frac{dX(t)}{X(t)}=qdt \Rightarrow X\left(t\right)=X\left(0\right)e^{qt}=e^{qt}$.

Define $V\left(t\right)$ =value of shares invested in asset, which is equal to $X\left(t\right)S\left(t\right)$. Thus by Ito's lemma,

$$\frac{dV}{V}=\frac{dX}{X}+\frac{dS}{S}+\underbrace{\left(\frac{dX}{X}\right)\left(\frac{dS}{S}\right)}_{=0}$$

$$=qdt+\mu dt+\sigma dB=\left(\mu+q\right)dt+\sigma dB.$$

5. Black-Scholes-Merton Formula

Consider an European call option with maturity at time T and exercise price K. The underlying asset pays a constant dividend yield q with price S and satisfies,

$$\frac{dS}{S}=\mu dt+\sigma dB, \tag{5.1}$$

where μ is some general random process; σ is assumed to be constant; and B is a Brownian motion. In this economy there is a riskless asset that pays a constant continuously-compunded risk-free rate of return r. We also assume that there exists some stochastic discount factor (SDF) $\tilde{\phi}$ such that the price at date $t=0$ of any security that pays $\tilde{X}$ at date T is $E\left[\tilde{\phi}\tilde{X}\right]$. The call option pays $Max\left(0,S\left(T\right)-K\right)$ at date T.

Definition. (**Digital option**) Let $\tilde{y}=\begin{cases}1 & if \quad S\left(T\right)>K \ \ (in\ the\ money)\\ 0 & o.w.\end{cases}$, so that the (digital or binary) call option pays $S\left(T\right)\tilde{y}-K\tilde{y}$.

That is, the payoff of a replicating portfolio comprising a long position in the underlying risky asset and a short position in the riskless asset with value $Y(t)$ at date $t \leq T$. Note that the replicating portfolio pays no-dividend, in the sense that there is no cash distributed outside the portfolio. Moreover, recall that the fundamental pricing formula is $Y(0) = num(0) E^{num}\left[\frac{Y(T)}{num(T)}\right]$.

We seek to answer the following:

1) What is the value at time $t = 0$ of getting $K\tilde{y}$ at date T?

Using the fundamental asset pricing formula and R as the numeraire gives $Y(0) = 1 \times E^R\left[\frac{Y(T)}{e^{rT}}\right] = e^{-rT} K E^R[\tilde{y}] = e^{-rT} K prob^R (S(T) \geq K)$, by definition of probability.

2) What is the value at time $t = 0$ of getting $S(T)\tilde{y}$ at date T?

Using once more the fundamental asset pricing formula and choosing V now as the numeraire such that $V(0) = S(0)$ and $V(T) = e^{qT} S(T) \Rightarrow Y(0) = S(0) E^V\left[\frac{S(T)\tilde{y}}{e^{qT} S(T)}\right] = e^{-qT} S(0) E^V[\tilde{y}] = e^{-qT} S(0) prob^V [S(T) \geq K]$.

Thus, the value at date $t = 0$ of the European call is,

$$e^{-qT} S(0) prob^V [S(T) \geq K] - e^{-rT} K prob^R [S(T) \geq K]. \tag{5.2}$$

3) Calculate $prob^R [S(T) \geq K]$.

Recall $prob^R [S(T) \geq K] = E\left[1_{\{S(T) \geq K\}} \tilde{\phi} e^{rt}\right]$. Use the fact that $\frac{V(t)}{e^{rt}}$ is a martingale under $prob^R$. Also, $V(t) = e^{qt} S(t)$ and $\frac{dV}{V} = (\mu + q) dt + \sigma dB$. Set $Z(t) = \frac{V(t)}{e^{rt}} \Rightarrow \frac{dZ}{Z} = \frac{dV}{V} - rdt$ by Ito's lemma, so that $Z(t) = (\mu + q - r) dt + \sigma dB$, where B is a Brownian motion under actual probability. Define $dB^* = dB + \frac{\mu+q-r}{\sigma} dt$ as the Brownian motion under the risk-neutral probability. Then $Z(t) = \sigma dB^*$.

By Levy's theorem a random process is a Brownian motion if it is a continuous martingale with quadratic variation equal to T. Recall $\frac{dS}{S} = \mu dt + \sigma dB$, substituting $\frac{dS}{S} = \mu dt + \sigma\left(dB^* - \frac{\mu+q-r}{\sigma} dt\right) = (r - q) dt + \sigma dB^* \Rightarrow dlogS(t) = \left(r - q - \frac{1}{2}\sigma^2\right) dt + \sigma dB^* \Rightarrow logS(T) = logS(0) + \left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma B^*(T)$. Notice that $S(T) \geq K \Leftrightarrow logS(T) \geq logK \Leftrightarrow logS(0) + \left(r - q - \frac{1}{2}\sigma^2\right) T + \sigma B^*(T) \geq logK \Leftrightarrow log\left(\frac{S(0)}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right) T \geq -\sigma B^*(T)$

$\Leftrightarrow \frac{log\left(\frac{S(0)}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} \geq -\frac{B^*(T)}{\sqrt{T}}$, where $\frac{B^*(T)}{\sqrt{T}} \sim N(0,1)$. Thus,

$$prob^R [S(T) \geq K] = N\left(\underbrace{\frac{log\left(\frac{S(0)}{K}\right) + \left(r - q - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}}_{=d_2}\right) = N(d_2).$$

4) Calculate $prob^V [S(T) \geq K]$.

We use the fact that $\frac{e^{rt}}{V(t)}$ is a martingale under $prob^V$. Set $Z(t) = \frac{e^{rt}}{V(t)} = \frac{R(t)}{V(t)} \Rightarrow \frac{dZ}{Z} = rdt - \frac{dV}{V} + \left(\frac{dV}{V}\right)^2$, by Ito's lemma. So,

$Z(t) = (r - \mu - q) dt - \sigma dB + \sigma^2 dt = \left(r - \mu - q + \sigma^2\right) dt - \sigma dB$. Re-arranging terms $Z(t) = -\sigma\left(dB - \frac{r-q-\mu+\sigma^2}{\sigma} dt\right) = -\sigma dB^*$ where $dB^* = dB + \frac{\mu+q-r+\sigma^2}{\sigma} dt$.

By Levy's theorem B^* is a Brownian motion under $prob^V$. Recall $\frac{dS}{S} = \mu dt + \sigma dB$, substituting $\frac{dS}{S} = \mu dt + \sigma\left(dB^* - \frac{\mu+q-r+\sigma^2}{\sigma} dt\right) = \left(r - q + \sigma^2\right) dt + \sigma dB^* \Rightarrow$

$dlogS\left(t\right)=\left(r-q+\frac{1}{2}\sigma^{2}\right)dt+\sigma dB^{*}\Rightarrow logS\left(T\right)=logS\left(0\right)+\left(r-q+\frac{1}{2}\sigma^{2}\right)T+\sigma B^{*}\left(T\right)$.

Notice that $S\left(T\right)\geq K\Leftrightarrow logS\left(T\right)\geq logK\Leftrightarrow logS\left(0\right)+\left(r-q+\frac{1}{2}\sigma^{2}\right)T+\sigma B^{*}\left(T\right)\geq logK\Leftrightarrow log\left(\frac{S(0)}{K}\right)+\left(r-q+\frac{1}{2}\sigma^{2}\right)T\geq-\sigma B^{*}\left(T\right)$

$\Leftrightarrow\frac{log\left(\frac{S(0)}{K}\right)+\left(r-q+\frac{1}{2}\sigma^{2}\right)T}{\sigma\sqrt{T}}\geq-\frac{B^{*}(T)}{\sqrt{T}}$, where $\frac{B^{*}(T)}{\sqrt{T}}\sim N\left(0,1\right)$. Thus,

$$prob^{V}\left[S\left(T\right)\geq K\right]=N\left(\underbrace{\frac{log\left(\frac{S(0)}{K}\right)+\left(r-q+\frac{1}{2}\sigma^{2}\right)T}{\sigma\sqrt{T}}}_{=d_{1}}\right)=N\left(d_{1}\right),$$

Substituting these probabilities in equation (5.2) gives the BSM formula for the value at time $t=0$ of an European call option,

$$e^{-qT}S\left(0\right)N\left(d_{1}\right)-e^{-rT}KN\left(d_{2}\right), \tag{5.3}$$

where $d_{2}=d_{1}-\sigma\sqrt{T}$.

6. Greeks, Delta Hedging, and the BSM equilibrium PDE

Definition 6. (**Greeks**) The derivatives of the BSM pricing formula are known as "Greeks". The most important greek is delta $\Delta=\frac{\partial C}{\partial S}$ that measures the sensitivity of the option value to changes in the value of the underlying asset. Gamma $\Gamma=\frac{\partial^{2}C}{\partial S^{2}}$ measures the sensitivity of delta to changes in the value of the underlying asset. Theta $\Theta=-\frac{\partial C}{\partial t}$ measures the sensitivity of the option value w.r.t. time with negative sign as time-to-maturity decreases.

6.1. **Delta hedging.** Let $dS=\mu Sdt+\sigma SdB$ where both the drift and difussion are functions of S and t, and B is some Brownian motion. Also, $C\left(S,T\right)$ is the value at date $t\leq T$ of an European call option on the underlying S with maturity T. To compute the change in value of the contingent claim we replace T with $T-t$ and $S\left(0\right)$ with S in the BSM formula and use Ito's lemma,

$$dC=\frac{\partial C}{\partial t}dt+\frac{\partial C}{\partial S}dS+\frac{1}{2}\frac{\partial^{2}C}{\partial S^{2}}\left(dS\right)^{2}. \tag{6.1}$$

Suppose now that you short the call, hold Δ shares of the underlying stock and $C\left(S,T\right)-\Delta S$ in the risk free asset. Let W be the value of such hedging portfolio. Thus, the change of the value of the hedging portfolio at instant dt is,

$$dW=-dC+\left(C\left(S,T\right)-\Delta S\right)rdt+\Delta\left(dS+qSdt\right),$$

$$=-\frac{\partial C}{\partial t}dt-\frac{\partial C}{\partial S}dS-\frac{1}{2}\frac{\partial^{2}C}{\partial S^{2}}\left(dS\right)^{2}+\left(C\left(S,T\right)-\Delta S\right)rdt+\Delta dS+\Delta qSdt,$$

$$=\underbrace{-\Theta dt}_{(+)}\underbrace{-\frac{1}{2}\Gamma\sigma^{2}S^{2}dt}_{(-)}+\Delta qSdt+\left(C\left(S,T\right)-\Delta S\right)rdt. \tag{6.2}$$

Note that this hedge eliminates all exposure to changes in the price of the underlying. On the other hand, the value of the portfolio increases as time passes with rate $-\Theta$. Moreover, as the payoff from the long position is always below the payoff from the short position by convexity, then as we move far from $S\left(t\right)$ we are worse because we hold a short position in convexity or "Gamma". Finally, note that the perfect delta hedging strategy entails $dW = 0$ and rearranging terms leads to the BSM equilibrium partial differential equation (PDE),

$$\frac{1}{2}\Gamma\sigma^2 S^2 dt + \Delta S\left(r-q\right)dt + \left(\Theta - Cr\right)dt = \frac{1}{2}\Gamma\sigma^2 S^2 + \Delta S\left(r-q\right) - Cr + \Theta = 0. \tag{6.3}$$

7. Empirical Estimation of Option Pricing Models

After the 1987 crash it was evident that the implied volatilities from S&P 500 index options exhibited a smile or smirk not consistent with the BSM model. Call options that were deep in the money have higher implied volatilities than those near the money. This pattern suggests that the risk-neutral return distribution is not log-normal but exhibits fat tails and is negatively skewed.

7.1. **Heston model.**

$$\frac{dS}{S} = \mu\left(S,\nu\right)dt + \sqrt{\nu}dB_S + dZ_S,$$

$$d\nu = \kappa\left(\bar{\nu} - \nu\right)dt + \sigma\sqrt{\nu}\left(\rho dB_S + \sqrt{1-\rho^2}dB_\nu\right) + dZ_\nu,$$

where $\mu\left(S,\nu\right)$ is the stock price and volatility dependent drift; $\left(B_S, B_\nu\right)$ is a vector of independent Brownian motions in $\mathbb{R}^2$; $\rho \in (0,1)$ is the constant correlation coefficient; and the processes $\left(Z_S, Z_\nu\right)$ are jumps with intensities (i.e., arrival rates) λ_S, λ_ν, respectively. Note that the amplitudes of the jumps are random. Heston (1993) considered the special case $\left(\lambda_S, \lambda_\nu\right) = 0$, consistent with the non-arbitrage opportunities assumption. He showed that the square-root difussion assumption for stochastic volatility allows computation of risk-neutral probabilities by Fourier inversion of the characteristic function as it entails a known closed form for it.

7.2. **Parametric estimation strategies.**

$$\underset{\theta\in\Theta}{argmin}\left\{\epsilon_{i,t}^2\right\},\ where$$

$$\epsilon_{i,t} \equiv \frac{C_{i,t}}{S_t} - O\left(X_t, T_i - t, k_{i,t}\left|\theta\right.\right),$$

where i identifies each option in the sample with different strike prices and maturities; and θ is the vector of parameters of the option pricing model used e.g., some version of Heston's model. The vector $\hat{\theta}$ can be estimated using a cross-section of strike prices and maturities for a given day "recalibrating" the model every day.

An alternative estimation strategy is to fixed the parameters over time, group options according to their moneyness and maturity, and adopt the following error components structure,

$$\epsilon_{i,t} = \epsilon_{I,t} + \sigma_I\eta_{i,t} \quad for\ i \in G\left(I,t\right)$$

$$\epsilon_{I,t} = \rho_I\epsilon_{I,t-1} + v_{I,t},$$

where $G\left(I,t\right)$ is the set of observations for group I at date t; $v_{I,t}$ is a zero-mean white noise common to all options within group I, possibly correlated across groups; and $\eta_{i,t} \sim N\left(0,1\right)$ is uncorrelated with $v_{I,t}$. The error components structure can be estimated using a Kalman filter and generalized least-squares (GLS).

Another approach is to use the simulated-moments estimator (SME) of Gallant and Tauchen (1996), using short-term at the money (ATM) call option prices i.e., $k_t \in [0.97, 1.03]$. Under this approach the econometrician has to make assumptions about the data generation process (DGP) of strike prices and contract maturities of the ATM calls. Other strategy that have been used in the empirical literature is the implied-state general-method-of-moments (IS-GMM),

$$\theta_T = \underset{\theta\in\Theta}{argmin} G_T(\theta)^T \mathbb{W}_{\mathbb{T}} G_T(\theta),$$

$$G_T(\theta) = \frac{1}{T}\sum_{t\leq T} h\left(s_t, \nu_t^\theta \,|\theta\right),$$

where $\mathbb{W}_{\mathbb{T}}$ is a positive semi-definite distance matrix; h is the moment generating function; $s_t$are log-deflated (excess) returns; and ν_t^θ is the date-t option-implied volatility.

Notice that when using GMM or ML to estimate option pricing models the econometrician has to choose the maturity of the observed option as close as possible to some given maturity (e.g., 30 days), subject to the constraint that the option price is not too far out of the money. This allows standard asymptotic theory to be directly applicable to assess the properties of the estimator.

Finally, when one or more state variables are assumed to be latent or the stochastic volatility follows a multifactor process with few degrees of freedom, then one can use a Monte Carlo Markov Chain (MCMC) estimation procedure. In general the empirical analyses showed that,

1. the inclusion of stochastic volatility improves substantially option pricing minimizing out-of-sample pricing errors, and
2. the inclusion of jumps result in a small improvement in option pricing unless the investment horizon is sufficiently long.

References

[Gallant and Tauchen (1996) Econometric Theory]
[Heston (1993) Review of Financial Studies]