LECTURE NOTES 2

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I. Arrow-Debreu (Pure-Exchange) Economy

In a timeless pure exchange economy under certainty, choices over commodities are based on the utility or felicity score final consumption provides to the economic agent. Consider now an exchange economy extending through a finite sequence of dates $t = 1, \ldots, T$. In moving to a multi-period economy, the availability of commodities through time is not certain as it may depend on the realization of some state of nature. Also, equal quantities of the commodity may result in different felicity scores depending on the realized state of nature. Consequently, commodities become *contingent*.

Uncertainty or more properly risk is defined by a finite and exhaustive set of mutually exclusive states of nature $s = 1, \ldots, S$. Note that time plays no explicit role, but states of nature unfold over time. The risky economy is represented by a history of observable events (sets of states) from date $t = 1$ to date $t = T$ i.e., a partition $\mathcal{S}_t \in S$. We assume the sequence of partitions to be monotone and non-decreasing in fineness, mesh, or norm, that is $\mathcal{S}_{t+1}$ is as fine as $\mathcal{S}_t$. A partition $\mathcal{S}$ is said to be as fine as partition $\mathcal{S}'$ and $\mathcal{S}'$ as coarse as $\mathcal{S}$, if for every $\mathcal{A}' \in \mathcal{S}'$ and $\mathcal{A} \in \mathcal{S}$ either $\mathcal{A} \subset \mathcal{A}'$ or $\mathcal{A} \cap \mathcal{A}' = \varnothing$. Intuition: The filtration (time indexed set of algebras) generated by $\mathcal{S}_{t+1}$ is larger (carries more information) than the one generated by $\mathcal{S}_t$. Note that $\mathcal{S}_t = \{S\}$ forms an *event-tree*.

At each date t there is a finite set of commodities $c = 1, \ldots, C$ and a finite set of intertemporal traders indexed by $i \in I$.

Definition 1. (Contingent commodity) For every commodity $c = 1, \ldots, C$, intertemporal trader $i \in I$, and state of nature $s = 1, \ldots, S$, a unit of (state)-contingent commodity x_{csi} is the ith trader's claim to receive a unit of commodity c if and only if state of nature s occurs such that,

$$\sum_{i=1}^{I} x_{csi} = x_{cs}. \tag{1}$$

Accordingly, the ith trader's (state)-contingent commodity vector is

$x_i = \left(\underbrace{x_{11i}, \ldots, x_{C1i}}_{State\ 1}, \ldots, \underbrace{x_{1Si}, \ldots, x_{CSi}}_{State\ S} \right) \in \mathbb{R}^{CS}$ (*Note that a negative entry represents an obligation to deliver the commodity*). The (state)-contingent commodity vector can be viewed as a collection of C random variables with support $(x_{c1}, \ldots, x_{cS}) \forall c$. Let the vector of exogenously determined contingent endowments

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(i.e., potential consumption) be $\omega_i \in \mathbb{R}_+^{CS}$ at date $t = 0$. In this context, intertemporal traders attach a (subjective or objective?) probability π_{si} to the occurrance of each state of nature s.

Weak assumption: Let $U_i(x_i)$ be some utility function representing the ith intertemporal trader's rational preference relation $\succeq_i$ on $x_i \in \mathbb{R}^{CS}$.

Theorem. ***(Utility representation theorem)*** *The preference relation $\succeq$ has an expected utility representation if for every $s \in S$ there is a (Bernoulli state-dependent) function $u_s : \mathbb{R}_+ \to \mathbb{R}$ such that for any $x, x' \in \mathbb{R}^{CS}$, $x \succeq x'$ if and only if $\sum_s \pi_s u_s(x) \geq \sum_s \pi_s u_s(x')$.*

Strong assumption: At date $t = 0$ and before the resolution of uncertainty forward markets for each contingent commodity are opened in the economy with prices $\bar{p}_{cs}$. We assume perfectly competitive markets, that is every trader has negligible market power and consequently act as price takers.

Definition 2. (State contingent economy) A state contingent or stochastic economy is characterized by $\mathbb{E} = \left\{I, S, \{\succeq_i\}_{i\in I}, \{\omega_i\}_{i\in I}\right\}$.

Then,

Definition 3. (Arrow-Debreu equilibrium) The pair that includes an allocation x^*_{csi} and price vector $\bar{p}_{cs} \in \mathbb{R}_+^{CS}$ is an Arrow-Debreu equilibrium if and only if for every $i \in I$, x^*_{csi} maximizes $\succeq_i$ subject to the budget constraint $B_i = \left\{x_{csi} \in \mathbb{R}^{CS} \middle| \sum_{c=1}^{C}\sum_{s=1}^{S} \bar{p}_{cs} \cdot x_{csi} = \sum_{c=1}^{C}\sum_{s=1}^{S} \bar{p}_{cs} \cdot \omega_{csi}\right\}$, and all opened markets clear $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i$ satisfying Walras' law.

Note. The solution involves the utilization of the same optimization tools of standard consumption theory under certainty. For example, one can derive the indirect utility function, or use duality and derive the expenditure function, write the Lagrangian function and obtain a Slutsky decomposition of the intertemporal demands from first order necessary conditions.

Proposition. *Let $\succeq_i$ be strictly convex and assume non-satiation at each date-event, then given the price system $\bar{p}_{cs}$ the optimal risk-bearing allocation x^*_{ics} can be attained by trading contingent commodities in perfectly competitive forward markets at time $t = 0$ so uncertainty resolves ex-ante.*

Crucial assumptions: Local non-satiation at each date-event implies that intertemporal traders attach positive probabilities to all events. Local non-satiation plus strict convexity of the preferences set guarantees the existence of a unique global interior solution, which can be found applying Kuhn-Tucker necessary and sufficient conditions.

Note. This is a very abstract economy. Trade across states is physically impossible. Implicitly all individuals have the same (symmetric) information on the event-tree and this is common knowledge. Although static in nature, the one-shot Arrow-Debreu economy still entails implicit dynamics as uncertainty is resolved ex-ante but trade is implemented ex-post (after state s realizes).

Theorem. ***(First fundamental theorem of welfare economics)*** *A competitive Arrow-Debreu equilibrium is Pareto optimal.*

Theorem. ***(Second fundamental theorem of welfare economics)*** *Given a Pareto optimal allocation, then there exists an Arrow-Debreu equilibrium that supports that allocation.*

Claim. If x^*_{csi} is Pareto optimal ex-ante, then x^*_{csi} is Pareto optimal ex-post.

Intuition: We assume now that uncertainty resolves at period $t = 1$ in a two-period economy. When period $t = 1$ arrives, state of nature s reveals, trade is executed, and every intertemporal trader receives the optimal bundle x^*_{csi}. We assume also that before the actual consumption of x^*_{csi} a spot market opens at time $t = 1$ for ex-post trade. Would there be any incentive to trade in this market? The answer is **NO**. If there were any potential gains from ex-post trading the Arrow-Debreu equilibrium allocation can't be Pareto optimal, contradicting the first theorem of welfare economics. In other words ex-ante Pareto optimality implies ex-post Pareto optimality and thus there should be no ex-post trading.

II. Radner Sequential Trading (Pure Exchange) Economy

We now allow for sequential trading in the spot markets. We assume that at $t = 0$ intertemporal traders have expectations about the spot prices of the contingent commodities at time $t = 1$ for each possible state $s \in S$. We denote the price vector that prevails in state of nature s as $p_s \in \mathbb{R}^C_+$, and the overall spot price system by $p = (p_1, \dots, p_S) \in \mathbb{R}^{CS}_+$. Furthermore, we assume that at date $t = 0$ the forward market is only opened for the contingent commodity with label 1. Think of this commodity as the store of value or purchasing power. We denote $q = (q_1, \dots, q_S) \in \mathbb{R}^S_+$ as the state dependent price vector of this commodity at time $t = 0$.

At time $t = 0$, each intertemporal trader observes $q \in \mathbb{R}^S_+$ and $p = (p_1, \dots, p_S) \in \mathbb{R}^{CS}_+$. Each intertemporal trader i formulates and buys a trading (equivalently a portfolio) forward (saving/lending) plan for the store of value commodity $z_i = (z_{1i}, \dots, z_{Si}) \in \mathbb{R}^S$ at time $t = 0$, and chooses a set of consumption plans $x_i = (x_{1i}, \dots, x_{Si}) \in \mathbb{R}^{CS}_+$ to buy at time $t = 1$ once uncertainty is revealed and contracts regarding z are executed. Hence, the behavioral problem of the intertemporal trader i is,

$$\underset{\{z_i, x_i\}}{Max}\, U_i(x_i) \tag{2}$$

$$s.t. \quad \textstyle\sum_s \quad q_s \cdot z_{si} \leq 0$$

$$p_s \cdot x_{si} \quad \leq \quad p_s \cdot \omega_{si} + p_{1s} \cdot z_{si} \forall s \in S$$

Notice that we have assumed zero initial endowments for the contingent commodities (i.e., negative sign of the budget constraint).However, we are not imposing any restriction on the sign or magnitude of z_{si}. If $z_{si} \leq -\omega_{1si}$ then we can think of the trader selling short the store of value commodity (consuming/borrowing) at time $t = 0$. This possibility though is limited indirectly by the fact that consumption and therefore ex-post wealth should be non-negative in every state s. The state-dependent budget at time $t = 1$ is composed of the market value of the endowment given the price vector p plus the market value of the store of value commodity bought or sold at time $t = 0$.

Key condition (Rational expectations hypothesis): We impose the condition that expectations must be self-fulfilled or rational; that is intertemporal traders are endowed with perfect foresight as their expectations of prices that will clear the spot markets for each state s are correct i.e., they do actually clear the markets at time $t = 1$ when state s occurs. (Note the implicit assumption in this condition of symmetric information across traders).

Definition 4. (Radner equilibrium) The forward price vector $q \in \mathbb{R}^S_+$, the (expected) price vector $p_s \in \mathbb{R}^C_+$ for every s at time $t = 0$, plus the trading plan $z^*_i \in \mathbb{R}^S$ at time $t = 0$, and consumption plans $x^*_i \in \mathbb{R}^{CS}_+$at time 1 that solve the behavioral problem (2) for all $i \in I$, and satisfy the market clearing conditions $\sum_i z^*_{si} \leq 0$ and $\sum_i x^*_{si} \leq \sum_i \omega^*_{si}$ for every state s is a Radner equilibrium.

Note. In the more realistic Radner economy as trade takes place sequentially the intertemporal traders face a sequence of contingent budget sets at every date-event. In this regard, the equilibrium is viewed as a pseudo equilibrium of expectations, prices, and plans in the economy. If the set of all traders' expectations are based on current market conditions we say that the pseudo equilibrium is a correct expectations equilibrium allowing divergence of opinion about the future across traders. When (subjective) opinions about the future coincide with the actual (objective) outcome then we say that the equilibrium is a rational expectations equilibrium. When the set of all traders' expectations are based on past and current market conditions the pseudo equilibrium is a temporary equilibrium with adaptive expectations.

Convention. Commodity 1 can be seen as the numeraire of the Radner economy.

Intuition: Note that all the budget constraints are homogeneous of degree zero with respect to prices i.e., the budget sets remain the same if the price of one commodity in each date-state is arbitrarily normalized to equal 1. Lets say that we chose to normalize the first commodity (as we actually did). Then for every state s a unit of commodity 1 pays off 1 dollar in state s and $p_{1s} = 1$. For time $t = 0$ we can convene that $q_1 = 1$ and maybe $\sum_s q_s = 1$. Thus, in the Radner economy the store of value commodity is the numeraire (the standard by which the values of the rest of the contingent commodities can be measured). We left deliberately open the issue of what type of commodity actually represents the numeraire, besides the fact that it has to be suitable to be stored (not perishable).

Proposition. *(i) If the pair that includes the allocation $x^* \in \mathbb{R}^{CSI}$ and price vector $\bar{p} \in \mathbb{R}^{CS}_{++}$ constitutes an Arrow-Debreu equilibrium, then there exists a price vector $q \in \mathbb{R}^S_{++}$ and trading plan $z^* \in \mathbb{R}^{SI}$ for the numeraire such that $\{x^*, z^*, q, \bar{p}\}$, constitute a Radner equilibrium.*

(ii) Conversely, if the plans x^*, z^* and prices q, p constitute a Radner equilibrium, then there exists a vector of multipliers $\mu = (\mu_1, \ldots, \mu_S) \in \mathbb{R}^S_{++}$ such that the allocation $x^*$and price vector $p \cdot \mu \in \mathbb{R}^{CS}_{++}$ constitute an Arrow-Debreu equilibrium (Think of the multipliers as shadow state prices).

1. Asset Markets. Notice that at the end we convene that the shopping basket of the ith intertemporal trader at time $t = 0$ collapses to a contingent claim to receive/deliver z_{1i} units of commodity 1 if state 1 realizes, z_{2i} units of commodity 1 if state 2 realizes,..., and z_{Si} units of commodity 1 if state S realizes. The total cost of the trading-consumption plan is $q_1 z_{1i} + \cdots + q_S z_{Si}$. Realistically, transfering wealth across states is hard in physical markets. Hence, we define a unit of an asset or market security as a title or claim to receive/deliver either physical goods or dollars at time $t = 1$ in amounts that depend on which state s realizes. The payoffs of these assets at time $t = 1$ are labeled as returns. If the returns are denominated in physical goods the asset is said to be real. If the returns are denominated in paper money the asset is said to be financial. For simplicity, we assume that payoffs are denominated in units of the numeraire. Since there is a single good (the numeraire) in each state available at the two dates, we can include date $t = 0$ as state $s = 0$ and re-define the commodity space as $\mathbb{R}^{S+1}$. Moreover, given non-satiation it must be that $x_i \in \mathbb{R}_+^{S+1}$.

We assume competitive asset markets, that is every trader has negligible market power and consequently act as a price taker, there are no position limits or short sale constraints, there are no transaction costs such as bid-ask spreads, and no indivisibilities such as a minimum amount to trade. The financial contracts have objectively stated conditions, which are the same for all traders, and are assumed to be perfectly monitored and enforced.

Definition 5. (Asset) With some abuse of notation we define an asset as a claim at time t in event A that entitles the ith trader to receive (a duty to deliver if the amount is negative) an amount $z_{t\tau}(A, A')$ of numeraire in the market at time τ contingent on event A'. Note that $\tau = t = 0$ defines the spot market, whereas $\tau > t = 1$ defines the forward market.

Definition 6. (Asset return) The amount of numeraire received/delivered at time $t = 1$ is the return R_s of the asset at date $t = 1$ if state $s \in S$ occurs. An asset therefore can be characterized by its return vector $R = (R_1, \dots, R_S) \in \mathbb{R}^S$.

Definition 7. (Riskless asset) A riskless asset is the one that entitles a unit of the numeraire independently of what state realizes i.e., $R_f = \left(1, 1, \dots, \underbrace{1}_{s}, \dots, 1\right)$.

Definition 8. (Pure asset) A pure asset or primitive Arrow-Debreu security of the sth type is a claim to receive a unit of the numeraire if state s realizes and nothing otherwise i.e, $R = \left(0, 0, \dots, \underbrace{1}_{s}, \dots, 0\right) \Leftrightarrow z_s = 1$.

Intuition: This concept will allow the logical decomposition of assets into portfolios of pure securities.

For each pair of dates t, τ such that $\tau > t$, and each (state)-contingent commodity c, there is a structure of K assets freely traded in the markets at time t with vector of returns $R_k \in \mathbb{R}^K$. The asset structure corresponds to a given family of events $A_{t\tau}$, which is either empty or equal to some partition of the state space S. In the later case, note that S_τ must be as fine (less coarse) than $A_{t\tau}$. In words, if at date t each trader can trade to deliver at date τ contingent on event A', then at a later date $\tau > v > t$ the trader can do the same. This is the no regret principle (dynamic consistency condition).

Definition 9. (Allowable asset) We say that an asset is allowable if for $A \in S_t$, $A' \in A_{t\tau}$, $A' \subseteq A$ there is some positive constant n such that $z_{t\tau}(A, A') \leq n$.

Definition 10. (Asset portfolio) A trade plan or portfolio z for the ith intertemporal trader is the (column) vector of assets $z_i = \left(z_i^1, \ldots, z_i^K\right)^T \in \mathbb{R}^K$ of k allowable assets with cost $q^1 z_i^1 + \cdots + q^K z_i^K$ at time $t = 0$ and return $\left(z_i^1 r_1 + \cdots + z_i^K r_K\right) \in \mathbb{R}^S$ at time $t = 1$.

Definition 11. (Radner equilibrium with assets) The price vector $q = \left(q^1, \ldots, q^K\right) \in \mathbb{R}_+^K$ for the K assets traded at time $t = 0$, the vector of (expected) spot prices $p_s = (p_{1s}, \ldots, p_{Cs}) \in \mathbb{R}_+^C$ for every s at time $t = 1$, the portfolio strategy at time $t = 0$ $z_i^* \in \mathbb{R}^K$, and the consumption plans $x_i^* \in \mathbb{R}_+^{S+1}$ at time $t = 1$ that solve the behavioral problem of each ith intertemporal trader,

$$\underset{\{z_i, x_i\}}{Max}\, U_i\left(x_i\right) \tag{3}$$

$$s.t. \quad \textstyle\sum_k \quad q^k z_i^k \leq 0$$

$$p_s x_{si} \quad \leq \quad p_s \omega_{si} + \sum_k p_{1s} z_i^k R_{sk} \; \forall s$$

and satisfy the market clearing condition $\sum_k z_i^{k*} \leq 0$ for every asset k and state s, and allocation feasibility constraint $\sum_i x_{si}^* \leq \sum_i \omega_{si}^*$ is a Radner equilibrium. Recall that $p_{1s} = 1$. The budget constraint represents the wealth of trader i at state s composed of the market values of her endowment and investment portfolio.

Note. The Radner equilibrium of the economy $\mathbb{E}\left(\succeq, \omega, R\right)$ with asset markets seems to be (formally) equivalent to the one with contingent commodities $\mathbb{E}\left(\succeq, \omega, 1_S\right)$ where 1_S denotes the identity matrix with size $S \times S$. Buying one unit of asset z_i^k with vector return R_k and price q^k should be equivalent to buy a (state)-contingent trading-consumption plan z_i with unit return and price vector q. At this point, the relevant question is if there exists some price vector q that makes buying asset k equivalent to buy the trading-consumption plan z_i for all $i \in I$? Also, are there enough assets in the asset structure K to reproduce any plausible contingent trading-consumption plan z?

Remark. (*Two-fund separation*) In exchange economies with separable-homothetic preferences, if the utility functions are weakly separable across states (i.e., the aggregate spot market demands for the c contingent commodities in each state s depend only on the aggregate income in each state) and identically homothetic within states then the equilibrium vector of (expected) spot prices can be obtained independently of (before) the equilibrium analysis of the financial markets is carried out (we can fix these prices and proceed with the analysis in the financial markets like we did in the Lecture notes #1).

Definition 12. (Return matrix) The return matrix R of an asset structure is defined as an $S \times K$ matrix with kth column equal to the return vector of the kth asset, and generic sk entry R_{sk} (the return of asset k in state s),

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1K} \\ R_{21} & R_{22} & \cdots & R_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ R_{S1} & R_{S2} & \cdots & R_{SK} \end{bmatrix}.$$

Let W be the matrix of asset payoffs $W = W\left(q^k, R\right) = \begin{bmatrix} -q^k \\ R \end{bmatrix}$. Then the budget set can be re-written as $B\left(q^k, \omega_i, R\right) = \left\{x_i \in \mathbb{R}_+^{CS} \left| x_i - \omega_i = W \cdot z_i,\ z_i \in \mathbb{R}^K \right.\right\}$. The intertemporal trader i chooses portfolio z_i such that $x_i - \omega_i = W \cdot z_i$ (i.e., to finance x_i).

Important Implication for Empirical Applications: The knowledge of R suffices to place significant restrictions on the asset price vector $q^k = \left(q^1, \ldots, q^K\right) \in \mathbb{R}^K$ that may arise in a Radner equilibrium.

Definition 13. **(Arbitrage portfolio)** the portfolio strategy $z_i \in \mathbb{R}^K$ is an arbitrage given the price vector q^k if and only if $Wz_i > 0\ \forall i \in I$. That is, a free lunch.

Definition 14. **(Arbitrage-free price vector)** If there is no arbitrage portfolio given the price vector q^k, then we say that the price vector q^k is arbitrage-free with respect to R.

Note. The absence of arbitrage is equivalent to the fundamental principle of no free lunch in economics -what Koopmans (1951) called as the impossibility of the land of Cockaigne. Notice that is a more primitive concept than market equilibrium, in the sense that it is independent of the characteristics of the intertemporal traders, and depends only on the technology of the asset structure. It is a stronger condition than the law of one price. That is, the law of one price is implied by the absence of arbitrage opportunities but not the other way. It seems that the arbitrage-free price vector concept first appeared in a paper by von Neumann in 1937.

Definition 15. **(Market subspace)** The set of all wealth transfers or income w_i that can be obtained by trading assets in the markets is a (linear) subspace of $\mathbb{R}^{S+1}$ denoted by $\langle W \rangle$. That is, the set of all opportunities for risk-sharing offered by the financial markets,

$$\langle W \rangle = \left\{w \in \mathbb{R}^{S+1} \left| w = Wz,\ z \in \mathbb{R}^K \right.\right\}. \tag{4}$$

Remark. The absence of arbitrage opportunities can be represented geometrically as the crossing of the (linear) market subspace $\langle W \rangle$ with the non-negative orthant $\mathbb{R}_+^{S+1}$ at the origin i.e., $\langle W \rangle \cap \mathbb{R}_+^{S+1} = \{0\}$. That is, wealth cannot be obtained in some state without giving it up in another state.

Theorem. ***(First fundamental theorem of asset pricing)*** *Assume that the vector of asset prices q^k is arbitrage-free. Then for every column vector $q^k \in \mathbb{R}^K$ of asset prices arising in a Radner equilibrium we can find a row vector of multipliers $\mu = (\mu_1, \ldots, \mu_S) \geq 0$ such that $q_k = \sum_s \mu_s R_{sk}$ for all k ($q^T = \mu \cdot R$ in matrix notation).*

Proof. Note that an arbitrage-free price row vector q^k implies $\langle W \rangle \cap \mathbb{R}_+^{S+1} = \{0\}$. Since both $\langle W \rangle$ and $\mathbb{R}_+^{S+1}$ are convex sets and the origin belongs to $\langle W \rangle$, we apply the separating hyperplane theorem to obtain a non-zero vector $\mu' = (\mu_1', \ldots, \mu_S')$ such that $\mu' \cdot w \leq 0$ for any $w \in \langle W \rangle$ and $\mu' \cdot w \geq 0$ for any $w \in \mathbb{R}_+^S$. Note that it must be that $\mu' \geq 0$. Moreover, because $w \in \langle W \rangle \Rightarrow -w \in \langle W \rangle$, it follows that $\mu' \cdot w = 0$ for any $w \in \langle W \rangle$.

Note now that if q^T is not proportional to $\mu' \cdot R \in \mathbb{R}^K$ then we can find a $z \in \mathbb{R}^K$such that $q \cdot z = 0$ and $\mu' \cdot Rz > 0$ (i.e., an arbitrage portfolio). But $w = Rz$

and $\mu' \cdot w \neq 0$, which can't be. Therefore, it must be that in an arbitrage-free market $q^T = \alpha\mu' \cdot R$ for some real number $\alpha > 0$. Finally, if $\mu = \alpha\mu'$ then $q^T = \mu \cdot R$ as required. □

Remark. The pricing function $q^T = \mu \cdot R$ or system of K present-value equations with S μ_s unknowns, is not unique as we can rotate the hyperplane orthogonal to the state price vector around the linear market subspace as long as does not cut into the positive orthant of free lunches. By doing so, we can define a whole range of pricing functions.

Remark. Geometrically the existence of a positive vector of state prices satisfy the condition $\langle W\rangle^{\perp} \cap \mathbb{R}^{S+1}_{++} \neq \varnothing$.

Definition 16. (Market completeness I - easy definition but only true for the (one-commodity) two period economy) An asset structure with an $S \times K$ return matrix R is complete if the rank of $R = S$. That is, if there is some subset of S assets that spans the partition S_τ i.e., $S_\tau = A_{t\tau}$. By span we mean the creation of assets from a linear combination of pure Arrow-Debreu securities.

Note. Recall that the rank of a matrix is the maximum number of linearly independent rows in the matrix.

Definition 17. (Market completeness II - rigorous general definition) Let the market subspace $\langle W\rangle$ be arbitrage free. If the market subspace has maximal dimension S then the markets are complete, otherwise they are said to be incomplete i.e., $\mathbb{R}^{S+1} = \langle W\rangle \oplus \langle W\rangle^{\perp} \Rightarrow S+1 = dim\,\langle W\rangle + dim\,\langle W\rangle^{\perp} \Rightarrow dim\,\langle W\rangle \leq S+1-1 = S$.

Proposition. *Suppose that the asset structure is complete. Then:*

(i) If $x^* \in \mathbb{R}^{(S+1)I}$ and $(p_1, \ldots, p_S) \in \mathbb{R}^{S+1}$ constitute an Arrow-Debreu equilibrium, then there exist an asset price vector $q^k \in \mathbb{R}^K$ and portfolio plan $z^* \in \mathbb{R}^{KI}$ such that the optimal consumption plans, portfolio plans, vector of asset prices, and spot prices constitute a Radner equilibrium.

(ii) Conversely, if the consumption plans $x^* \in \mathbb{R}^{(S+1)I}$, portfolio plan $z^* \in \mathbb{R}^{KI}$, and price vectors $q^k \in \mathbb{R}^K$, $(p_1, \ldots, p_S) \in \mathbb{R}^{S+1}$ constitute a Radner equilibrium, then there exists a unique vector of multipliers $(\mu_1, \ldots, \mu_S) \in \mathbb{R}^S$ such that the optimal consumption plans, and the price vector $(\mu_1 p_1, \ldots, \mu_S p_S) \in \mathbb{R}^{S+1}$ constitute an Arrow-Debreu equilibrium.

Theorem. ***(Second fundamental theorem of asset pricing)*** *The vector of multipliers $\mu = (\mu_1, \ldots, \mu_S) \geq 0$ such that $q^k = \sum_s \mu_s R_{sk}$ for all k ($q^T = \mu \cdot R$ in matrix notation) is unique if the market is complete.*

Proof. From assuming that the market subspace $\langle W\rangle$ is arbitrage-free it follows that it is complete if and only if it is a hyperplane, meaning that its orthogonal subspace is one-dimensional, a condition that clearly implies the uniqueness of the state price vector as required. □

Intuition: By definition of market completeness, the opportunities for wealth transfers across states are the greatest when markets are complete as $dim\,\langle W\rangle = S$. On the other hand, this means that the (potential) differences of opinion among agents about present values are the smallest as it must be that $dim\,\langle W\rangle^{\perp} = 1$, and every $i \in I$ coincides in their shadow state prices $\mu_1 = \cdots = \mu_I = \mu$. On the other hand, when markets are incomplete opportunities are not that great as $dim \langle W \rangle < S$ and the intertemporal traders' opinions on the shadow prices will diverge as $dim \langle W \rangle^{\perp} = S - K + 1 > 1$ i.e., trading in the market no longer forces the traders' state price vector to coincide.

Proposition. *Suppose that the consumption plans $x^* \in \mathbb{R}^{(S+1)I}$, portfolio plans $z^* \in \mathbb{R}^{KI}$, and vector of prices $q \in \mathbb{R}^K$, $(p_1, \ldots, p_S) \in \mathbb{R}^{S+1}$ constitute a Radner equilibrium with an asset structure with return matrix R. Let R' be the $S \times K'$ return matrix of a second alternative asset structure. If the market subspace $\langle w' \rangle$ spanned by R'= market subspace $\langle w \rangle$spanned by R, then $x^* \in \mathbb{R}^{(S+1)I}$ is still the Radner equilibrium of the economy with the second asset structure.*

Definition 18. (Redundant asset) We say that an asset is redundant if its deletion does not affect the market subspace $\langle w \rangle$. That is, the return vector of a redundant asset is a linear combination of the return vectors of the remaining assets i.e., a replicating portfolio.

Example. (The Binomial Model)

The asset structure in the (one-period) economy is composed of:

1) a riskless asset with constant rate of return $\overline{r}$ continuously compounded (think of this return as the risk free rate),

2) a risky asset with initial price S_0 and price $\{S_d, S_u\}$ at time $\tau = 1$,

(A1) $S_u > S_d$,

(A2) no dividends,

Hence we define u as the up state and d as the down state,

3) a call option on the risky asset with strike price equal to K and payoffs,

$C_u = Max\,(0, S_u - K)$ in the up state, and

$C_d = Max\,(0, S_d - K)$in the down state,

(A3) $\frac{S_u}{S_0} > e^{\overline{r}\tau} > \frac{S_d}{S_0}$. No free lunches in the economy **(No-arbitrage principle)**. If the return on the stock were greater than the risk-free rate in both states, then we can buy an infinite amount of the stock on margin. Conversely, if the return on the stock were less than the risk-free rate in both states, then we can short an infinite amount of the stock and put the proceeds in the riskless asset.

Are markets complete in this economy?

Is the call option a redundant asset?

YES and YES. The rank of the return matrix $R = 2 = \#$ states in the economy. Why? because the call option is a redundant asset by definition: its return is a linear combination of the return vectors of the remaining assets and its deletion does not affect the market subspace $\langle w \rangle$. We show that the return vector of the call option can be mimic by a replicating portfolio conformed by the riskless and risky asset.

Find the arbitrage-free price C_0.

Define $\delta = \frac{C_u - C_d}{S_u - S_d} \Rightarrow \delta\,(S_u - S_d) = C_u - C_d \Rightarrow \delta S_u - C_u = \delta S_d - C_d$

Trading plan = Buy δ shares of the risky asset at time $t = 0$ on margin. That is, borrow $e^{-\overline{r}\tau}\,(\delta S_u - C_u)$.

At date T either we have (value of the delta shares - dollars we owe),

$\delta S_u - (\delta S_u - C_u) = C_u$ if the up state occurs, or

$\delta S_d - (\delta S_d - C_d) = C_d$ if the down state occurs.

The arbitrage free price of a call option is the one that makes $\delta S_0 = e^{-\overline{r}\tau}\,(\delta S_u - C_u)$ i.e., cost of delta shares should be equal to the amount of dollars borrowed.

Define the state prices $\pi_u = \frac{S_0 - e^{-\overline{r}\tau} S_d}{S_u - S_d}$ and $\pi_d = \frac{e^{-\overline{r}\tau} S_u - S_0}{S_u - S_d}$. That is π_u pays 1 if the up state realizes and 0 otherwise, and π_d pays 1 is the down state realizes and 0 otherwise (Do you see it?).[1] These are prices of pure Arrow-Debreu securities by definition. Given market completeness we can decompose the asset structure as a system of linear combinations of pure Arrow-Debreu securities, Assuming no arbitrage opportunities, there should exist positive state prices and the price of any security is the sum across states of its payoff times the state price:

$C_0 = \pi_u C_u + \pi_d C_d$,
$S_0 = \pi_u S_u + \pi_d S_d$, and
$1 = \pi_u e^{\overline{r}\tau} + \pi_d e^{\overline{r}\tau}$.

The next step in the analysis is to manipulate the state prices to obtain expectations. Define the (objective) probabilities p_u= probability of up state, and p_d= probability of down state. Note that the measure of the expectation is based on the risky asset.

Define the state price densities as $\phi_u = \frac{\pi_u}{p_u}$ and $\phi_d = \frac{\pi_d}{p_d}$.

Solving for the state prices and substituting the resulting values in the asset structure gives,

$C_0 = \phi_u p_u C_u + \phi_d p_d C_d$,
$S_0 = \phi_u p_u S_u + \phi_d p_d S_d$, and
$1 = \phi_u p_u e^{\overline{r}\tau} + \phi_d p_d e^{\overline{r}\tau}$.

By definition of expected value,

$C_0 = E\left[\tilde{\phi}\tilde{C}\right]$,
$S_0 = E\left[\tilde{\phi}\tilde{S}\right]$, and
$1 = E\left[\tilde{\phi} e^{\overline{r}\tau}\right]$.

Finally, by properties of expected values,

$\Rightarrow 1 = E\left[\tilde{\phi}\frac{\tilde{C}}{C_0}\right]$,
$\Rightarrow 1 = E\left[\tilde{\phi}\frac{\tilde{S}}{S_0}\right]$, and
$1 = E\left[\tilde{\phi} e^{\overline{r}\tau}\right]$.

Note that $\tilde{\phi}$ is the stochastic discount factor or pricing kernel. Recall that arbitrage pricing pins down to find a suitable specification for the stochastic discount factor or pricing kernel. Moreover, the no-arbitrage principle is the sufficient assumption to obtain this result as guaranteed by the fact that $\pi_u > 0$ and $\pi_d > 0$ (First fundamental theorem of asset pricing).

Alternatively, we can define expectations on the risk neutral measure. That is the one based on the risk free asset. Thus, we define risk-neutral probabilities $q_u = e^{\overline{r}\tau}\pi_u$ and $q_d = e^{\overline{r}\tau}\pi_d$ (a risk-neutral measure - the certainty equivalents (CEs) measure of microceconomics). Does the multipliers of the Arrow-Debreu/Radner economies ring a bell? Solving for the state prices and substituting the resulting equations in the asset structure gives,

$C_0 = q_u e^{-\overline{r}\tau} C_u + q_d e^{-\overline{r}\tau} C_d = e^{-\overline{r}\tau} E^{\mathbb{Q}}\left[\tilde{C}\right]$
$S_0 = q_u e^{-\overline{r}\tau} S_u + q_d e^{-\overline{r}\tau} S_d = e^{-\overline{r}\tau} E^{\mathbb{Q}}\left[\tilde{S}\right]$ and

[1] *From* $1 = \frac{S_0 - e^{-\overline{r}\tau} S_d}{S_u - S_d} e^{\overline{r}\tau} + \frac{e^{-\overline{r}\tau} S_u - S_0}{S_u - S_d} e^{\overline{r}\tau} \Rightarrow 1)$ *up state* : $\frac{S_0 - 0}{S_u - 0} e^{\overline{r}\tau} + \frac{S_0 - S_0}{S_u - 0} e^{\overline{r}\tau} = \frac{S_0}{S_u} e^{\overline{r}\tau} + 0 = \frac{e^{-\overline{r}\tau} S_u}{S_u} e^{\overline{r}\tau} = 1;\ 2)$ *down state* : $\frac{S_0 - S_0}{0 - S_d} e^{\overline{r}\tau} + \frac{0 - S_0}{0 - S_d} e^{\overline{r}\tau} = 0 + \frac{-e^{-\overline{r}\tau} S_d}{-S_d} e^{\overline{r}\tau} = 1$.

$1 = q_u + q_d$,
where $E^{\mathbb{Q}}[\cdot]$ denotes the expectational operator under the risk-neutral measure. The subjective stochastic discount factor is unique because all intertemporal traders face the same riskless asset and the market is complete. By definition, if a stochastic process M satisfies the condition $M_t = E_t[M_\tau] \forall t, \tau$, then M is a martingale. Clearly $\tilde{C}, \tilde{S}$ are martingales. Risk-neutral pricing is equivalent to the martingale approach to pricing.

2. Incomplete Markets. What happens to our theoretical construction if $dim \langle W \rangle < S$?

The first consequence is to the Radner equilibrium. In incomplete markets wealth transfers across states are limited, and then there might be some loss of wealth and the Radner equilibrium most likely will not be Pareto optimal. In this case, there may exist some alternative reallocation of consumption that will make all intertemporal traders at least as well off, and at least one consumer strictly better off than the pseudo equilibrium allocation.

Important: This does not imply that a welfare authority equipped with a social function can achieve the Pareto optimum. An allocation that cannot be Pareto improved by such an authority is a constrained Pareto optimum.

Definition 19. (Constrained Pareto optimum) The asset allocation $z \in \mathbb{R}^{KI}$ is a constrained Pareto optimum if it is allowable (i.e., $\sum_i z_i < 0$) and if there is no other allowable asset allocation $z' \in \mathbb{R}^{KI}$ such that $U_i^*(z') \geq U_i^*(z) \forall i$ with at least one inequality strict.

Proposition. ***(Third theorem of welfare economics)*** *Suppose that there are two periods and only one consumption good in the second period. Then any Radner equilibrium is a constrained Pareto optimum in the sense that there is no possible redistribution of assets in the first period that leaves every trader as well off and at least one trader strictly better off than before.*

This is Arrow's impossibility theorem extended to Radner economies.

Example. (The Trinomial Model)

The asset structure in the economy is composed of:

1) a riskless asset with constant return $\overline{r}$ continuously compounded (think of this return as the risk free rate),

2) a risky asset with initial price S_0 and price $\{S_d, S_m, S_u\}$ at time $\tau = 1$,

(A1) $S_u > S_m > S_d$,

(A2) no dividends,

Hence u is the up state, m is the middle, mean, median state, and d is the down state,

(A3) Either $\frac{S_u}{S_0} > e^{\overline{r}T} > \frac{S_m}{S_0} > \frac{S_d}{S_0}$ or $\frac{S_u}{S_0} > \frac{S_m}{S_0} > e^{\overline{r}T} > \frac{S_d}{S_0}$. No free lunchs in the economy **(No-arbitrage principle)**.

State prices π_u, π_m, and π_d must satisfy,

$S_0 = \pi_u S_u + \pi_m S_m + \pi_d S_d$, and

$1 = \pi_u e^{\overline{r}T} + \pi_m e^{\overline{r}T} + \pi_d e^{\overline{r}T}$.

In the binomial case, the system of equations can be solved uniquely for π_u and π_d. However, in the trinomial case we have only two equations and three unknowns. Thus, there exists many plausible solutions. We can take a particular solution and define risk-neutral probabilities as in the binomial case example. Alternatively, we can define probabilities using the stock as numeraire and value the call option. Anyways, the resulting value will depend on the particular solution we choose (π_u, π_m, π_d). There will be many arbitrage free values for the call option.

To see this, consider the replicating portfolio of a dollars invested in the riskless asset and b dollars invested in the stock. The value of the portfolio at time τ will be $ae^{\bar{r}\tau} + b\frac{S_x}{S_0}$ where $x = \{u, m, d\}$. To replicate a call option with strike price K we need,

$ae^{\bar{r}\tau} + b\frac{S_u}{S_0} = Max\left(0, S_u - K\right)$, and

$ae^{\bar{r}\tau} + b\frac{S_m}{S_0} = Max\left(0, S_m - K\right)$, and

$ae^{\bar{r}\tau} + b\frac{S_d}{S_0} = Max\left(0, S_d - K\right)$.

These is a system of three equations and two unknowns. For any strike price between S_u and S_d none of the equations is redundant and the system has no solution. As the rank of the return matrix $R <\#$ states, the market is incomplete.

Is the call option still a redundant asset? **NO.**

Moreover, we cannot price the call option using arbitrage pricing. If we pick a particular solution (π_u, π_m, π_d) and assume that the market uses this solution to price the call option it will be just one of many plausible ad-hoc solutions. Equivalently, we can assume that the market uses a particular set of risk neutral probabilities (q_u, q_m, q_d). In order to avoid arbitrary solutions, we would have to go back and assume something about the preferences and endowments of the intertemporal traders or RA and get an equilibrium solution. In other words, still we can use equilibrium theory to price the call option!

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