LECTURE NOTES 14

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1. Order Driven Market Models with Strategic Informed Traders

1.1. **The market as an efficient aggregator.**

(A.1) Assume I equal traders with forecast $f_i = V + e_i$, where V is the fundamental (unknown) value of the security and e_i is the ith trader forecast error such that $E[e_i] = 0$ i.e., f_i is an unbiased estimator.

(A.2) Each trader's position (net demand) in the security is $D_i = a(f_i - p)$ for some constant $a > 0$. If $f_i > p$ (*undervalued security*) $\Rightarrow D_i > 0 \Rightarrow$ long position; If $f_i < p$ (*overvalued security*) $\Rightarrow D_i < 0 \Rightarrow$ short position.

(A.3) Short sales are allowed so that the security is in zero net supply (simplifies the arithmetic).

So,

$$\sum_{i=1}^{I} D_i = \sum_i a(f_i - p) = a\sum_i f_i - Iap = 0,$$

$$\Rightarrow Iap = a\sum_i f_i,$$

$$\Rightarrow p = I^{-1}\sum_i f_i, \tag{1.1}$$

where the market price is the average of the individual forecasts. Substituting f_i into (1) gives,

$$p = I^{-1}\sum_i (V + e_i),$$

$$= V + I^{-1}\sum_i e_i,$$

$$= V + e_M, \tag{1.2}$$

where $e_M = I^{-1}\sum_i e_i$ is the average forecast error. If individual forecast errors are independent, then by the law of large numbers $e_M \to 0$ as $I \to +\infty$ and $p \to V$. If all traders have a common (and hidden) signal V then this information will be reflected in the market price and we say that markets are strong efficient.

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1.2. **Kyle (1985) model.**

(A.1) Single period model. Three type of traders in a market for a single security: a market maker; noise traders; and an insider.

(A.2) At the beginning of each period, traders trade in the risky asset that has a random end of period liquidation value $\tilde{v} \sim N(p_0, \Sigma_0)$.

(A.3) Noise traders submit a market order to buy $\tilde{u}$ shares of the asset, where $\tilde{u} \sim N\left(0, \sigma_u^2\right)$ and both $\tilde{u}$ and $\tilde{v}$ are independently distributed. We assume that noise traders are subject to exogenous shocks to their wealth and rebalance their portfolios accordingly.

(A.4) The single risk neutral insider knows with certainty the realized end of period value of the risky security $\tilde{v}$ (but not $\tilde{u}$) and submits a market order of size x that maximizes her expected end of period profit. This assumption can be weakened assuming some degree of uncertainty on $\tilde{v}$, which has to be less than the uncertainty of the market maker; and/or assuming the submission of a limit order instead of a market order that is dependent on the equilibrium market price (Kyle, 1989).

(A.5) The single risk neutral market maker (e.g., a specialist) observes the total order flow $y = x + \tilde{u}$, but is not able to distinguish what part is informed trading and what part is not (i.e., traders are anonymous). After observing the order flow, the market maker sets the price p and takes the position $-(x + \tilde{u})$ to clear the market. Note that market making is a perfectly competitive business, so the end of period profit of the market maker is expected to be zero.

We need to consider only the behavior of the market maker and the insider:

A) The price set by the market maker satisfies,

$$p = E\left[\tilde{v}\,|(\tilde{u} + x)\right].$$

The pricing rule of the market maker is a function of the order flow,

$$p = P(\tilde{u} + x).$$

B) The insider chooses to maximize her end of period profit $\tilde{\pi}$ given her knowledge of $\tilde{v}$ and the pricing rule of the market maker, which is common knowledge,

$$\underset{\{x\}}{max} E\left[\tilde{\pi}\,|v|\right] = \underset{\{x\}}{max} E\left[(v - P(\tilde{u} + x))\, x\,|v|\right].$$

Consequently, the equilibrium in this model consists of the pricing rule chosen by the market maker and the trading strategy chosen by the insider such that both the insider maximizes expected profit and the market maker breakevens.

Suppose an affine pricing rule of the form,

$$P(y) = \mu + \lambda y.$$

Thus,

$$\underset{\{x\}}{max} E\left[(v - \mu - \lambda(\tilde{u} + x))\, x\,|v|\right] = \underset{\{x\}}{max}\,(v - \mu - \lambda x)\, x,$$

since $E[\tilde{u}] = 0$. The F.O.N.C. is,

$$\frac{\partial \pi}{\partial x} \equiv v - \mu - 2\lambda x = 0,$$

and the solution to the insider's problem is,

$$x = \alpha + \beta v, \tag{1.3}$$

where $\alpha = -\frac{\mu}{2\lambda}$ and $\beta = \frac{1}{2\lambda}$.

For the market maker the best possible estimate of $E\left[\tilde{v}\,|(\tilde{u}+x)\right]$ is the MLE in the sense that attains maximum efficiency i.e., minimum variance unbiased estimator. Substituting (3) into the total order flow equation we get $y = \tilde{u} + \alpha + \beta\tilde{v}$, which is jointly normally distributed with $\tilde{v}$. Because of this assumption, the MLE is linear in y and equivalent to OLS. That is, the best linear unbiased (BLUE) estimator is the one that minimizes,

$$E\left[(\tilde{v} - P(y))^2\right] = E\left[(\tilde{v} - \mu - \lambda y)^2\right] = E\left[(\tilde{v} - \mu - \lambda(\tilde{u} + \alpha + \beta\tilde{v}))^2\right].$$

And the optimal pricing rule is the one that solves,

$$\underset{\{\mu,\lambda\}}{min} E\left[(\tilde{v}(1-\lambda\beta) - \lambda\tilde{u} - \mu - \lambda\alpha)^2\right].$$

Notice that $E[\tilde{v}] = p_0$, $E\left[(\tilde{v} - p_0)^2\right] = \Sigma_0$, $E[\tilde{u}] = 0$, $E\left[\tilde{u}^2\right] = \sigma_u^2$, and $E[\tilde{u}\tilde{v}] = 0$. Thus,

$$\underset{\{\mu,\lambda\}}{min}\ (1-\lambda\beta)^2\left(\Sigma_0 + p_0^2\right) + (\mu + \lambda\alpha)^2 + \lambda^2\sigma_u^2 - 2(\mu + \lambda\alpha)(1-\lambda\beta)\,p_0. \tag{1.4}$$

The F.O.N.C.s with respect to μ and λ are,

$$\mu = -\lambda\alpha + p_0(1-\lambda\beta),\ and$$

$$-2\beta(1-\lambda\beta)\left(\Sigma_0 + p_0^2\right) + 2\alpha(\mu + \lambda\alpha) + 2\lambda\sigma_u^2 - 2p_0\left[-\beta(\mu+\lambda\alpha) + \alpha(1-\lambda\beta)\right] = 0.$$

Substituting the first F.O.N.C. $\mu + \lambda\alpha = p_0(1-\lambda\beta)$ into the second F.O.N.C. gives,

$$\lambda = \frac{\beta\Sigma_0}{\beta^2\Sigma_0 + \sigma_u^2}.$$

Substituting the definitions of α and β in both F.O.N.C.s leads to,

$$\mu = p_0,\ and \tag{1.5}$$

$$\lambda = \frac{1}{2}\frac{\sqrt{\Sigma_0}}{\sigma_u}, \tag{1.6}$$

with equilibrium price,

$$p = p_0 + \frac{1}{2}\frac{\sqrt{\Sigma_0}}{\sigma_u}(\tilde{u} + x), \tag{1.7}$$

and equilibrium insider's strategy,

$$x = \frac{\sigma_u}{\sqrt{\Sigma_0}}(\tilde{v} - p_0). \tag{1.8}$$

Notice that the greater is the noise to signal ration, the larger is the magnitude of the order submitted by the insider given some deviation of v from its unconditional mean. Hence, the insider trades more actively on his private information the greater is the "camouflage" provided by noise trading. This is so because noise trading makes difficult for the market maker to extract the "signal" from the order flow, and consequently the insider can exploit her information advantage to increase her profit. Substituting equation (8) into (7) gives,

$$p = p_0 + \frac{1}{2}\frac{\sqrt{\Sigma_0}}{\sigma_u}\tilde{u} + \frac{1}{2}(\tilde{v} - p_0),$$

$$= \frac{1}{2}\tilde{v} + noise, \tag{1.9}$$

where $noise = \frac{1}{2}\left(p_0 + \frac{\sqrt{\Sigma_0}}{\sigma_u}\tilde{u}\right)$. Consequently, even in the best case scenario that $noise \to 0$, the market price will only convey half the insider's private information $p \to \frac{1}{2}\tilde{v}$. That is, the price is not be fully revealing and we say that markets are semi-strong efficient.

Finally, note that λ is the amount that the market maker raises the price when the total order flow y goes up by one unit. Hence, the amount of order flow necessary to raise the price by \$1 equals $\frac{1}{\lambda} = 2\frac{\sigma_u}{\sqrt{\Sigma_0}}$, which is a measure of the "depth" of the market, a second dimension of market liquidity. Notice that the higher is the noise to signal ratio, the deeper or more liquid is the market and the trade of the insider has a lower price impact inducing her to trade more without revealing her information advantage.

References

[Kyle (1985) Econometrica]
[Kyle (1989) Review of Financial Studies]