LECTURE NOTES 12

ARIEL M. VIALE

1. Market Microstructure

1.1. **Introduction.** Standard asset pricing theory assumes that all investors hold the same information set. However, in realistic economies different investors hold different information sets. Prices play a dual role:

- they constitute an index of scarcity, and
- they convey information about investors' expectations about the value of assets.

Assuming rational expectations (REH), in equilibrium investors' subjective beliefs must correspond to the actual or objective probability distributions characterizing asset returns' state-space. The advantage of this hypothesis is that investors may infer useful information from publicly observable prices along with their private information.

Market microstructure focus on the role that the structure of the markets has in the revelation of information through the price discovery process. Market microstructure models can be classified along the following four dimensions:

- type of orders,
- sequence of moves by traders,
- price setting rules, and
- competitive versus strategic setting.

The classification is as follows...

(1) Simultaneous submission of demand schedules:
    - Competitive models.
    - Strategic share auctions.
(2) Sequential move models
    - Screening models.
    - Sequential trade models.
    - Strategic market order models.

1.2. **Simultaneous demand schedule models.**

1.2.1. *Competitive models.* Asymmetric information is introduced assuming that the market is formed by two groups of risk-averse investors: N informed and M uninformed investors, plus Z noise/liquidity traders. Each group of investors submit their whole demand schedules X_I, X and $\tilde{x}$, respectively. Recall that in competitive markets investors act as price takers i.e., they see their actions as negligible enough to influence prices so there is no need to act strategically. For seek of simplicity, assume that investors receive no endowment at period $t = 0$ and let the risk-free interest rate to be $r_f = 0$.

Date: 11/06/11 - Rev.: 11/25/13.

Each investor holds a common prior about the random future liquidation value $\tilde{\nu}$ of the risky asset,

$$\tilde{\nu} \sim N\left(\bar{\nu}, \sigma_{\nu}^{2}\right).$$

At time $t = 0$, the insiders receive a private signal $\tilde{S}^{i}$ on ν that is i.i.d. and drawn from,

$$\tilde{S}^{i}\left|\nu\right. \sim N\left(\nu, \sigma_{S}^{2}\right),$$

with mean ν and variance σ_{S}^{2}. This is common knowledge to all investors.

The prior for $\tilde{\nu}$ is,

$$g\left(\tilde{\nu}\right) = \frac{1}{\sqrt{2\pi}\sigma_{\nu}} exp\left[-\frac{1}{2\sigma_{\nu}^{2}}\left(\tilde{\nu} - \bar{\nu}\right)^{2}\right].$$

The conditional distribution of the signal is,

$$f\left(\tilde{S}\left|\nu\right.\right) = \frac{1}{\sqrt{2\pi}\sigma_{S}} exp\left[-\frac{1}{2\sigma_{S}^{2}}\left(\tilde{S} - \nu\right)^{2}\right].$$

Applying Bayes' rule to compute the posterior density function for $\tilde{\nu}$ given S leads to,

$$g\left(\tilde{\nu}\left|S\right.\right) = \frac{f\left(\tilde{S}\left|\nu\right.\right) g\left(\tilde{\nu}\right)}{\int f\left(\tilde{S}\left|\nu\right.\right) g\left(\tilde{\nu}\right) d\nu}.$$

We make use of the projection theorem for Normal distributions to obtain the conditional mean and variance,

$$E\left[\tilde{\nu}\left|S\right.\right] = E\left[\tilde{\nu}\right] + \frac{Cov\left(\tilde{\nu}, \tilde{S}\right)}{Var\left(\tilde{S}\right)}\left[\tilde{S} - E\left[\tilde{S}\right]\right],$$

and,

$$Var\left(\tilde{\nu}\left|S\right.\right) = \sigma_{\nu}^{2} - \frac{\left[Cov\left(\tilde{\nu}, \tilde{S}\right)\right]^{2}}{Var\left(\tilde{S}\right)}.$$

After some manipulation the posterior equals,

$$g\left(\tilde{\nu}\left|S\right.\right) \sim N\left(\frac{\tau_{S}}{\tau_{\nu} + \tau_{S}}S + \frac{\tau_{\nu}}{\tau_{S} + \tau_{\nu}}\bar{\nu}, \frac{1}{\tau_{\nu} + \tau_{S}}\right),$$

where $\tau_{\nu} = \sigma_{\nu}^{-2}$ and $\tau_{S} = \sigma_{S}^{-2}$ are the precision of $\tilde{\nu}$ and $\tilde{S}$, respectively. We write $\beta = \frac{\tau_{S}}{\tau_{\nu} + \tau_{S}}$ to finally get,

$$g\left(\tilde{\nu}\left|S\right.\right) \sim N\left(\beta S + \left(1 - \beta\right)\bar{\nu}, \left(1 - \beta\right)\sigma_{\nu}^{2}\right).$$

At time $t = 1$, trading takes place through risk-averse investors submiting limit orders and liquidity traders submiting market orders. Finally, at time $t = 2$, the value of the asset is revealed to all market players. Preferences are assumed to be represented by an exponential (CARA) utility function with equal coefficient of risk aversion A. The behavioral problem of each risk-averse investor is,

$$\underset{X}{Max}\left\{E\left[u\left(\tilde{\pi}\right) = -exp\left(-A\tilde{\pi}\right)\right]\right\},$$

where $\tilde{\pi} = X\left(\tilde{\nu} - P\right)$ is the end of period profit under price P with Normal distribution,

$$\tilde{\pi} \sim N\left(X\left(E\left[\tilde{\nu}\right] - P\right), X^{2}\sigma_{\nu}^{2}\right).$$

Exploiting the property of the moment generating function for a Normal random variable leads to,

$$\underset{X}{Max}\left\{-exp\left(-A\tilde{\pi}\right)\right\} = \underset{X}{Max}\left\{E\left[\tilde{\pi}\right] - \frac{A}{2}Var\left(\tilde{\pi}\right)\right\}.$$

The F.O.N.C. condition w.r.t. X gives the optimal demand for the risky asset,

$$X_I^* = \frac{E\left[\tilde{\nu}\left|S\right.\right] - P}{AVar\left[\tilde{\nu}\left|S\right.\right]}, \tag{1.1}$$

where $E\left[\tilde{\nu}\left|S\right.\right] = \beta S + (1-\beta)\,\bar{\nu}$ and $Var\left[\tilde{\nu}\left|S\right.\right] = (1-\beta)\,\sigma_\nu^2$.

1.2.2. *Naive expectations equilibrium (NEE).* Plugging X_I, X and $\tilde{x}$ into the market clearing condition $NX_I + MX + Z\tilde{x} = 0$ gives,

$$N\left(\frac{E\left[\tilde{\nu}\left|S\right.\right] - P}{AVar\left[\tilde{\nu}\left|S\right.\right]}\right) + M\left(\frac{E\left[\tilde{\nu}\right] - P}{AVar\left[\tilde{\nu}\right]}\right) + Z\tilde{x} = 0.$$

The three groups of investors come to the market and submit their respective demands to the Walrasian auctioneer who ensures that the market clears. Solving for P we obtain,

$$P = \mu_1 E\left[\tilde{\nu}\left|S\right.\right] + (1-\mu_1)\,E\left[\tilde{\nu}\right] + \mu_2\tilde{x},$$

where $\mu_1 = \frac{NVar[\tilde{\nu}]}{[NVar[\tilde{\nu}]+MVar[\tilde{\nu}|S]]}$; and $\mu_2 = \frac{AZVar[\tilde{\nu}]}{[NVar[\tilde{\nu}]+MVar[\tilde{\nu}|S]]}$. The equilibrium price is a combination of the weighted average of the expected liquidation values of insiders and uninformed risk-averse investors plus a risk premium for noise trading. Making the appropriate substitutions gives,

$$P = E\left[\tilde{\nu}\right] + \mu_1\frac{Cov\left(\tilde{\nu},\tilde{S}\right)}{Var\left(\tilde{S}\right)}\left[\tilde{S} - E\left[\tilde{S}\right]\right] + \mu_2\tilde{x}. \tag{1.2}$$

Notice that under this equilibrium uninformed investors only use their priors to update their expectations about the liquidation value of the risky asset in a naive way. That is, they do not use the information contained in the price to update their beliefs.

1.2.3. *Rational expectations equilibrium (REE).* We assume that,

$$\tilde{S} = \nu + \epsilon_S,$$

with $\tilde{S} \sim N\left(0,\sigma_S^2\right)$, $\epsilon_S \sim N\left(0,\sigma_\epsilon^2\right)$, and $\tilde{S}\perp\epsilon$. As a result, we can write $\tilde{\nu} \sim N\left(0,\sigma_S^2+\sigma_\epsilon^2\right)$, $E\left[\tilde{\nu}\left|S\right.\right] = S$, and $Var\left[\tilde{\nu}\left|S\right.\right] = \sigma_\epsilon^2$. Considering rational investors that use the market price to update their beliefs the demands of the uninformed and informed are respectively,

$$X^* = \frac{E\left[\tilde{\nu}\left|P\right.\right] - P}{AVar\left[\tilde{\nu}\left|P\right.\right]}, \tag{1.3}$$

and,

$$X_I^* = \frac{E\left[\tilde{\nu}\left|S,P\right.\right] - P}{AVar\left[\tilde{\nu}\left|S,P\right.\right]} = \frac{E\left[\tilde{\nu}\left|S\right.\right] - P}{AVar\left[\tilde{\nu}\left|S\right.\right]}. \tag{1.4}$$

Notice that the insiders' signal is now a sufficient statistic. That is, insiders can't learn anything from the market as the market price reflects both public information and all private signals, which by assumption are the same across insiders. Grossman (1986) solves the REE as follows:

(1) The uninformed investors make a conjecture about the equilibrium price of the form,

$$P = \alpha_1 \tilde{S} + \alpha_2 \tilde{x}.$$

(2) They use this conjecture to estimate $E[\tilde{\nu}\,|P]$ and $Var[\tilde{\nu}\,|P]$ and then determine their optimal demands X^*.

(3) Equilibrium is attained when the price that clears the market is a linear combination of both S and x. Recall that under RE in equilibrium the investors' subjective conjectures must coincide with the market realizations.

Thus, the equilibrium price is,

$$P = \mu_1 E[\tilde{\nu}\,|S] + (1-\mu_1) E[\tilde{\nu}] + \mu_2 \tilde{x},$$

$$= \mu_1 \left(\frac{\tau_S}{\tau_\nu + \tau_S} S + \frac{\tau_\nu}{\tau_S + \tau_\nu} \bar{\nu} \right) + (1-\mu_1)\bar{\nu} + \mu_2 \tilde{x}. \tag{1.5}$$

Note that the only unknown in equation (5) is the insiders' signal S. We differentiate two cases:

Case 1. Full revealing equilibrium

Case 2. In the case of an exogenous non-random supply x, the uninformed investor will infer the signal perfectly from the price P and update her beliefs consequently,

$$X = X_I^* = \frac{E[\tilde{\nu}\,|S] - P}{AVar[\tilde{\nu}\,|S]},$$

Case 3. with REE equilibrium,

$$P = E[\tilde{\nu}\,|S] - \frac{AVar[\tilde{\nu}\,|S]\, Zx}{N+M}. \tag{1.6}$$

Radner (1979) calls the linear REE equilibrium (6) a "full communication equilibrium". DeMarzo and Skiadas (1998) demonstrate that this equilibrium is quasi-complete as long as x is small. Note that CARA utility implies that investors' optimal demands are independent of income. Moreover, if the price reflects all available information i.e., is a sufficient statistic, then the private signal should play no role in the optimal investors' demands. The striking result, is that the price itself plays no role in the determination of investors' optimal demands. The reason for this is that the substitution effect is exactly offset by the information effect.

Definition 1. (Grossman-Stiglitz paradox) In a REE with endogenous information acquisition risk-averse investors have no incentive to collect costly information. However, if nobody gathers information then the price cannot reveal it and a competitive REE does not exist.

1.2.4. *Noisy REE.* The only uncertainty investors face in the REE model is the one pertaining the liquidation value ν. However, there are many uncertain factors that potentially may affect the equilibrium price, but not necessarily the hidden liquidation value. The simplest strategy to model additional uncertainty is to assume $\tilde{x} \sim N(0, \sigma_x^2)$. This makes the price only partially revealing because risk-averse uninformed investors will not be able to disentangle the price change due to noise trading from the change due to informed trading.

We assume that each uninformed trader conjectures that the other $(M-1)$ uninformed investors submit a downward-sloping demand function of the type,

$$X = -HP.$$

Substituting the demand of the insiders and the latter into the market clearing condition gives,

$$N\left(\frac{\tilde{S}-P}{A\sigma_{\epsilon}^{2}}\right) - (M-1)\,HP + X + Z\tilde{x} = 0.$$

Solving for the price leads to,

$$P = \lambda\left[\frac{N\tilde{S}}{A\sigma_{\epsilon}^{2}} + X + Z\tilde{x}\right],$$

with,

$$\lambda = \left[\frac{N}{A\sigma_{\epsilon}^{2}} + (M-1)\,H\right]^{-1},$$

which the rational uninformed uses to update her expectation about the liquidation value $\tilde{\nu}$. Define the residual signal extracted from the price as,

$$\tilde{\Theta} = \tilde{S} + \frac{A\sigma_{\epsilon}^{2}Z}{N}\tilde{x} = \gamma_1 P - \gamma_2 X = \Theta,$$

with,

$$\gamma_1 = \frac{N + A\sigma_{\epsilon}^{2}(M-1)\,H}{N} \; and \; \gamma_2 = \frac{A\sigma_{\epsilon}^{2}}{N}.$$

Then,

$$E\left[\tilde{\nu}\,|P\right] = E\left[\tilde{\nu}\,|\Theta\right] \; and \; Var\left[\tilde{\nu}\,|P\right] = Var\left[\tilde{\nu}\,|\Theta\right].$$

Using the projection theorem we compute,

$$E\left[\tilde{\nu}\,|\Theta\right] = \left(\frac{\sigma_S^2}{\sigma_S^2 + \frac{A^2\sigma_{\epsilon}^4 Z^2}{N^2}\sigma_x^2}\right)(\gamma_1 P - \gamma_2 X),$$

and,

$$Var\left[\tilde{\nu}\,|\Theta\right] = \sigma_S^2 + \sigma_{\epsilon}^2 - \frac{\sigma_S^4}{\sigma_S^2 + \frac{A^2\sigma_{\epsilon}^4 Z^2}{N^2}\sigma_x^2}.$$

Substituting the last expressions into the uninformed demand functions and solving for X gives,

$$X = -\left[\frac{1 - \frac{cov\left(\tilde{\nu,\tilde{\Theta}}\right)}{Var\left(\tilde{\Theta}\right)}\gamma_1}{AVar\left(\tilde{\nu}\,|\Theta\right) + \frac{cov\left(\tilde{\nu,\tilde{\Theta}}\right)}{Var\left(\tilde{\Theta}\right)}\gamma_2}\right]P. \tag{1.7}$$

The REH requires that the investors' subjective conjectures coincide with the market realizations. Thus, in equilibrium it must be,

$$H = \left[\frac{1 - \frac{cov\left(\tilde{\nu,\tilde{\Theta}}\right)}{Var\left(\tilde{\Theta}\right)}\gamma_1}{AVar\left(\tilde{\nu}\,|\Theta\right) + \frac{cov\left(\tilde{\nu,\tilde{\Theta}}\right)}{Var\left(\tilde{\Theta}\right)}\gamma_2}\right].$$

Substituting γ_1 and γ_2 gives,

$$H^* = \left[\frac{1 - \frac{cov\left(\nu, \tilde{\Theta}\right)}{Var\left(\tilde{\Theta}\right)}}{AVar\left(\tilde{\nu}\,|\Theta\right) + \frac{cov\left(\nu,\tilde{\Theta}\right)}{Var\left(\tilde{\Theta}\right)} \frac{MA\sigma_\epsilon^2}{N}} \right].$$

The noisy REE equilibrium is,

$$P^* = \left[\frac{N}{A\sigma_\epsilon^2} + MH^* \right]^{-1} \left[\frac{N}{A\sigma_\epsilon^2}\tilde{S} + Z\tilde{x} \right]. \tag{1.8}$$

In this setting an indicator of market liquidity is,

$$L = \left| \frac{dP}{dx} \right|^{-1} = \left[\frac{N}{A\sigma_\epsilon^2} + MH^* \right] = \frac{N}{A\sigma_\epsilon^2} + \frac{M\left(1 - \frac{cov\left(\nu,\tilde{\Theta}\right)}{Var\left(\tilde{\Theta}\right)}\right)}{AVar\left(\tilde{\nu}\,|\Theta\right) + \frac{MA\sigma_\epsilon^2 cov\left(\nu,\tilde{\Theta}\right)}{NVar\left(\tilde{\Theta}\right)}}. \tag{1.9}$$

An indicator of price volatility is,

$$Var\left[P^*\right] = (L)^{-2} \left[\frac{N^2}{A^2\sigma_\epsilon^4}\sigma_S^2 + Z^2\sigma_X^2 \right]. \tag{1.10}$$

And finally an indicator of informational efficiency is,

$$IE = \left(Var\left[\tilde{\nu}\,|\Theta\right]\right)^{-1} = \left(\sigma_S^2 + \sigma_\epsilon^2 - \frac{\sigma_S^4}{\sigma_S^2 + \frac{A^2\sigma_\epsilon^4 Z^2}{N^2}\sigma_x^2} \right)^{-1}. \tag{1.11}$$

Notice that L measures the inverse of the price impact of each noise trader's order. Clearly, the smaller the price impact, the greater the depth of the market, and hence the greater the liquidity. Moreover, the greater the depth of the market the lower is price volatility. Finally, the lower the conditional volatility of the liquidation value, the better is the quality of the equilibrium price as a vehicle of information about $\tilde{v}$.

The noisy REE setup was initiated by Grossman and Stiglitz (1980), Hellwig (1980), and Diamond and Verrechia (1981). Admati (1985) extends Hellwig's (1980) setting to multiple risky assets and infinitely many investors. Finally, Pfleiderer (1984) discusses how the change in the precision of the signal alters expected trading volume.

References

[Admati (1985) Econometrica]
[DeMarzo and Skiadas (1998) Journal of Economic Theory]
[Diamond and Verrechia (1981) Journal of Financial Economics]
[Grossman (1976) Journal of Finance]
[Grossman and Stiglitz (1980) American Economic Review]
[Hellwig (1980) Journal of Economic Theory]
[Pfleiderer (1984) Working paper]
[Radner (1979) Econometrica]