LECTURE NOTES 10

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1. Behavioral Asset Pricing

1.1. **Prospect theory based asset pricing model.** Barberis, Huang, and Santos (2001) assume a Lucas pure-exchange economy with three types of assets: a one-period riskless asset in zero net supply with return from date t to time $t+1$ $R_{f,t}$; a risky financial asset with random return R_{t+1} from date t to time $t+1$; and a non-financial asset (e.g., fixed capital like a house or human capital) with income Y_t. Consequently, consumption $C_t \equiv D_t + Y_t$ is not perfectly correlated with the stream of dividends from the financial asset D_t.

(A.1) Consumption and dividends follow a joint lognormal process,

$$ln\left(\frac{C_{t+1}^{agg}}{C_t^{agg}}\right) = g_C + \sigma_C \epsilon_{t+1},$$

$$ln\left(\frac{D_{t+1}}{D_t}\right) = g_D + \sigma_D \varepsilon_{t+1},$$

where the error terms are distributed as a multivariate Normal $\begin{pmatrix} \epsilon_t \\ \varepsilon_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$.

(A.2) Preferences are represented by,

$$E_0\left[\sum_{t=0}^{\infty}\left(\delta^t \frac{C_t^{\gamma}}{\gamma} + b_t \delta^{t+1} v\left(X_{t+1}, \pi_t, z_t\right)\right)\right],$$

where π denotes the one-period value of endowment/wealth W_t invested in the risky financial asset; $0 \neq \gamma < 1$; $\delta \in (0,1)$ is the subjective discount factor; $X_t \equiv \pi_{t-1}\left(R_t - R_{f,t-1}\right)$ is the total excess return over the risk-free return earned from holding the risky asset between date t and $t+1$; b_t is a scaling factor that makes the price-dividend ratio and risk premium stationary; $v(\cdot)$ is a piecewise linear value function that characterizes prospect theory cognitive biases effects on investors' utility independent of consumption (with a kink at the origin); and $z_t < (>)\, 1$ is a variable that denotes the historic return benchmark level given accumulated prior gains (losses) on the risky asset (i.e., the "reference point" of investors' framing process).

(A.5) $z_t = (1-\eta) + \eta z_{t-1}\frac{\bar{R}}{R_t}$, where $0 \leq \eta \leq 1$ is investor's memory effect from accumulated prior gains/losses; and $\bar{R}$ is the average risky asset return that makes $z_t = 1$ (break-even steady state). Notice, that if $\eta = 0$ (no memory) then $z_t = 1$ for all t. That is, the "reference point" tracks the value of the risky asset one-to-one representing a myopic investor. If

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$0 < \eta < 1$ then z_t adjusts sluggishly to the realized value of the risky asset given the weight of prior acumulated gains/losses. On the other hand, when $\eta = 1$ then the "reference point" moves too slowly to the realized value of the risky asset characterizing an investor with long memory. Furthermore, we differentiate two cases:

Case 1. If returns were relatively high last period i.e., $R_t > \bar{R}$ then $z_t < z_{t-1}$.

Case 2. If returns were relatively low last period i.e., $R_t < \bar{R}$ then $z_t > z_{t-1}$.

(A.6) For $z_t = 1$ the function $v(\cdot)$ displays pure loss aversion,

$$v(X_{t+1}, \pi_t, 1) = \begin{cases} X_{t+1} \; if \; X_{t+1} \geq 0 \\ \lambda X_{t+1} \; if \; X_{t+1} < 0 \end{cases}, for \; \lambda > 1.$$

(A.7) For $z_t \neq 1$ the value function $v(\cdot)$ displays the house money effect (i.e., the RA is less risk averse as previous gains cushion subsequent losses) with two cases:

Case 1. $z_t \leq 1$ i.e., the investor accumulated prior gains then,

$$v(X_{t+1}, \pi_t, z_t) = \begin{cases} X_{t+1} \; if \; R_{t+1} \geq z_t R_{f,t} \\ X_{t+1} + (\lambda - 1)\pi_t (R_{t+1} - z_t R_{f,t}) \; if \; R_{t+1} < z_t R_{f,t} \end{cases}, for \; \lambda > 1.$$

Case 2. $z_t > 1$ i.e., the investor accumulated prior losses then,

$$v(X_{t+1}, \pi_t, z_t) = \begin{cases} X_{t+1} \; if \; X_{t+1} \geq 0 \\ \lambda(z_t) X_{t+1} \; if \; X_{t+1} < 0 \end{cases}, for \; \lambda(z_t) = \lambda + k(z_t - 1), k > 0.$$

(A.8) $b_t = b_0 C_t^{\gamma-1}$ where $b_0 > 0$ is the investor's degree of framing.

Note that the state variables are W_t and z_t as the consumption-dividend joint process is i.i.d. To solve the intertemporal consumption-portfolio problem we follow Barberis and Huang (2004) and assume that wealth evolves as $W_{t+1} = (W_t - C_t)\left(\pi_t \tilde{R}_{t+1}\right) \equiv (W_t - C_t)\tilde{R}_{W,t+1}$ (i.e., non-financial income is totally consumed). Moreover, risky assets are seen as a "gamble" and we assume that the RA frames the gamble "narrowly" i.e., looking only at the stock market. Finally, the RA gets utility directly from the gamble and not indirectly from its contribution to total wealth. The RA has utility of the recursive form,

$$V_t = f(C_t, \mu(V_{t+1} | z_t)),$$

where $f(C, x) = [(1-\delta)C^\gamma + \delta x^\gamma]^{\frac{1}{\gamma}}$ is the aggregator function; and $\mu(\cdot)$ is an homogeneous of degree one certainty equivalent of the distribution of future utility V_{t+1} conditional on the reference point z_t. Hence, preferences are now modeled as,

(A.4)' $V_t = f(C_t, \mu(V_{t+1} | z_t) + b_0 E_t[\hat{v}(G_{t+1})]),$

where $G_{t+1} = \pi_t (W_t - C_t)(R_{t+1} - z_t)$.

Consider the special case $z_t = R_{f,t}$ (i.e., the reference point is the risk-free return). Thus, the HJB equation is,

(1.1)
$$J(W_t, z_t) = \underset{\{C_t, \pi_t\}}{Max} \left\{(1-\delta) C_t^\gamma + \delta \left[\mu\left(J(W_{t+1}, z_{t+1}) | z_t\right) + b_0 E_t[\hat{v}(G_{t+1})]\right]^\gamma\right\}^{\frac{1}{\gamma}}.$$

Write $J(W_t, z_t) = A(z_t) W_t \equiv A_t | W_t$. Thus $A_t W_t$ is equal to,

$$\underset{\{C_t, \pi_t\}}{Max} \left\{(1-\delta) C_t^\gamma + \delta (W_t - C_t)\left[\mu\left(A_{t+1}\pi_t R_{t+1} | z_t\right) + b_0 E_t[\hat{v}(\pi_t (R_{t+1} - R_{f,t}))]\right]^\gamma\right\}^{\frac{1}{\gamma}}.$$

As the consumption and portfolio decisions are separable, we can solve first the consumption problem and get the F.O.N.C. w.r.t. to C_t, which is the standard intertemporal envelope condition,

$$C_t^{\gamma} = J_W. \tag{1.2}$$

Define $\alpha_t \equiv \frac{C_t}{W_t}$ and write the Euler equilibrium equation,

$$1 = \left[\delta R_{f,t} E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]\right]\left[\delta E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{W,t+1}\right]\right]^{\frac{\gamma}{1-\gamma}}. \tag{1.3}$$

Solving the portfolio problem, the F.O.N.C. w.r.t. to π_t leads to the fundamental asset pricing equation for i risky financial assets,

$$\forall_i \; E_t\left[\hat{v}\left(R_{i,t+1} - R_{f,t}\right)\right] =$$

$$-\sum_i \pi_{i,t}^{*}\left(b_0 R_{f,t}\left(\frac{\delta}{1-\delta}\right)^{\frac{1}{1-\gamma}}\left(\frac{1-\alpha_t}{\alpha_t}\right)^{\frac{-\gamma}{1-\gamma}}\right)^{-1} \frac{E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(R_{W,t+1} - R_{f,t}\right)\right]}{E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]}. \tag{1.4}$$

Narrow framing of stocks can generate substantial equity premium potentially resolving Mehra & Prescott's equity risk premium puzzle. At the same time framing generates a sufficiently low risk-free return resolving the risk-free rate puzzle. The intuition is as follows: if the RA is more sensitive to losses (even small losses) than gains and gets utility directly from the value of the stock market, then she finds the stock market too risky and will only be willing to hold stocks for a relatively high average return. This model is also capable of generating persistence and reversal effects in stock returns. Finally, the RA's framing process is narrow in the sense that is only accounting for stock driven wealth, which represents a low percentage of total wealth.

Behavioral models attempt to provide a positive or descriptive theory of how investors "actually" behave in contrast with previous normative models that attempt to explain how rational investors "should" behave. There is experimental and empirical evidence of prospect theory cognitive biases like narrow framing and loss aversion. The house money effect does not seem to be supported by some recent empirical studies. One problem with the behavioral approach is that the Lucas economy with a behavioral RA is observational equivalent to a Lucas economy with a rational RA with state-dependent utility e.g., with habit consumption a la Constantinides/Campbell-Cochrane or a Bayesian RA that seeks to learn the current level of the dividend process a la Veronesi.

2. Robust Asset Pricing

2.1. Kogan and Wang (2003) CAPM under Knightian uncertainty.

(A.1) We assume a one-period economy. Consumption takes place at the end of period, and the RA is endowed with initial wealth $W_0 = 1$.

(A.2) There are N risky assets with vector of returns R and perfectly elastic supply, and one riskless asset in zero net supply with return r.

(A.3) The RA is assumed to be boundedly rational (i.e., like the econometrician of the conditional CAPM with a reduced information set). The

RA does not have perfect knowledge of the distribution of returns of the risky assets. Only knows that returns follow a joint Normal distribution,

$$f(R) = (2\pi)^{-\frac{1}{2}} |\Omega|^{-\frac{1}{2}} exp\left\{-\frac{1}{2}(R-\mu)^T \Omega^{-1}(R-\mu)\right\},$$

where $\mu = E[R]$; and $\Omega = E\left[(R-\mu)(R-\mu)^T\right]$. And knows Ω but not μ. This gives rise to "model uncertainty", "ambiguity", or uncertainty in the sense of Knight (1921) and Keynes (1921).

(A.4) Consequently, the RA has multi-prior expected utility preferences,

$$u(W, \mathcal{P}(P)) = \underset{\{Q \in \mathcal{P}(P)\}}{Min} \left\{E^Q[u(W)]\right\},$$

where E^Q denotes the expectation under the risk-neutral probability measure; $\mathcal{P}(P)$ is the set of probability measures that depend on some "reference" prior P and proxies the degree of model uncertainty perceived by the RA i.e., a larger set implies more Knightian uncertainty. The minimization operator captures the RA preference for uncertainty aversion or "robustness". Recall Ellsberg's paradox?

(A.5) Define $J_k = \{j_1, \ldots, j_{Nk}\}$ for $k = 1, \ldots, K$ subsets or "classes" of assets in the asset space $\{1, \ldots, N\}$ with Nk elements and not necessarily disjoint. Thus, we can represent the RA's information set derived from multiple sources of risk in the space of risky asset returns $R_{J_k} = (R_{J_1}, \ldots, R_{J_{Nk}})$. We assume that the investor has at least some information about each class/asset. Moreover, the probability distributions implied by the various sources of information not necessarily coincide with the marginal distributions under the "reference" model. The distribution function of R_{J_k} is,

$$(2\pi)^{-\frac{1}{2}} |\Omega_{J_k}|^{-\frac{1}{2}} exp\left\{-\frac{1}{2}(R_{J_k}-\hat{\mu}_{J_k})^T \Omega_{J_k}^{-1}(R_{J_k}-\hat{\mu}_{J_k})\right\},$$

where $\hat{\mu}_{J_k} = (\hat{\mu}_{J_1}, \ldots, \hat{\mu}_{J_{Nk}})$ is the "predicted" mean return under the "reference" model. If the likelihood ratio of the marginal distribution Q_{J_k} over P_{j_k} is $\mathcal{L}_{J_k}$, then the RA preferences can be now described by,

$$u(W, \nu) = \underset{\{\nu \in \mathcal{V}\}}{Min} \{E[\mathcal{L}u(W)]\},$$

where $\nu = \mu - \hat{\mu}$.

(A.6) Define $X = \pi^T R$ as the portfolio return with vector of weights π and Normally distributed. We define the uncertainty of X in the sense of Knight (1921) and Keynes (1921) as,

$$\Delta(\pi) = \underset{\{\nu\}}{Max} \left\{\pi^T \nu\right\},$$

subject to,

$$E[\mathcal{L}_{J_k} ln \mathcal{L}_{J_k}] = \frac{1}{2}\nu^T \hat{\Omega}_{J_k}^{-1} \nu \leq \eta_k,\ k = 1, \ldots, K,$$

where $\eta_k,\ k = 1, \ldots, K$, are the RA multiple confidence levels for the Nk likelihoods of the subsets J_k. The "uncertainty" metric is a dissimilarity measure known as the "Mahalanobis distance" between two random vectors Normally distributed. Note that distributions are close (bounded) to avoid infinite uncertainty aversion. For the special case that the covariance matrix is the identity matrix, the distance collapses to the familiar Euclidean distance. For the special case that the covariance matrix is diagonal, this distance collapses to the normalized Euclidean distance. The multiple confidence levels that the RA entertains about her "reference" model determine a confidence interval for the expected portfolio return $[-\Delta(\pi), \triangle(\pi)]$. Note that this metric is independent of the RA's utility, and is a convex and symmetric function of the portfolio weight, just like the variance. Hence, systematic uncertainty can be defined similar to systematic risk as the marginal contribution of the uncertainty of asset(class) i to the market portfolio uncertainty.

The RA portfolio choice problem can be stated as,

$$\underset{\{\pi\}}{Max}\,\underset{\{\nu\in\mathcal{V}\}}{Min}\,E\left[\mathcal{L}u(W)\right], \tag{2.1}$$

subject to the budget constraint,

$$W=\left[\pi^{T}(R-r\iota)+\iota+r\right]. \tag{2.2}$$

where ι denotes a vector of ones. The F.O.N.C. w.r.t. π is,

$$0=E\left[u'(W-\triangle(\pi))\,(R-r\iota-\nu(\pi))\right].$$

We define π_m as the market portfolio and $\triangle_m=\Delta(\pi_m)$ as the market's uncertainty. Thus, the stochastic discount factor or pricing kernel is,

$$M=\frac{u'(W-\triangle_m)}{E\left[u'(W-\triangle_m)\right]}. \tag{2.3}$$

In equilibrum risky asset must satisfy,

$$E[MR]=r\iota+\nu(\pi_m),$$

and the market return,

$$E[MR_m]=r+\triangle_m.$$

Applying Stein's lemma to both equations we find the fundamental asset pricing formula under Knightian uncertainty,

$$\mu-r\iota=\frac{E\left[u''(W-\triangle_m)\right]}{E\left[u'(W-\triangle_m)\right]}cov(R_m,R)+\nu(\pi_m),$$

$$\mu_m-r\iota=\underbrace{\frac{E\left[u''(W-\triangle_m)\right]}{E\left[u'(W-\triangle_m)\right]}}_{\lambda,risk\ premium}\sigma_m^2(\pi)+\underbrace{\triangle_m}_{\lambda_u,uncertainty\ premium}.$$

$$\Rightarrow(risk\ premium)=\lambda\beta+\lambda_u\beta_u, \tag{2.4}$$

where λ is the market price of risk; λ_u is the market price of uncertainty; β is market risk beta; and β_u is market uncertainty beta.

Clearly, the model is capable of resolving both the equity premium and the risk free puzzles. The problem with this model acknowledged by the authors is that (8) appears observationally equivalent to Merton's ICAPM.

References

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